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Theory of connections on graded principal bundles

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Abstract. The geometry of graded principal bundles is discussed in the framework of graded manifold theory of Kostant-Berezin-Leites. In particular, we prove that a graded principal bundle is globally trivial if and only if it admits a global graded section and, further, that the sheaf of vertical derivations on such a bundle coincides with the graded distribution induced by the action of the structure graded Lie group. This result leads to a natural definition of the graded connection in terms of graded distributions; its relation with Lie superalgebra-valued graded differential forms is also exhibited. Finally, we define the curvature for the graded connection; we prove that the curvature controls the involutivity of the horizontal graded distribution corresponding to the graded connection.

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1. Introduction

Graded manifolds are geometrical objects introduced by Kostant, [**15**] and Berezin, Leites, [**5**], as the natural mathematical tool in order to study supersymmetric problems. In particular, the formalism of graded manifolds provides the possibility to deal with classical dynamics of particles with spin. Indeed, it has been proved, [**6**], that the extension of ordinary phase spaces by anti-commuting variables yields the classical phase spaces of particles with spin. Thus, Kostant's article, [**15**], can be also considered as a treatment on super-Hamiltonian systems and on their geometrical prequantization. We mention [**12**], [**17**], [**18**], [**23**] as most important references related more or less to the global problems of Lagrangian supermechanics.

Our aim here is to investigate several aspects of the graded differential geometry, unexplored up today, and to use them in order to establish a theory of connections on graded principal bundles. This work is mainly motivated by the need to make clear why graded connection theory is convenient for the description of supersymmetric gauge theories, and how one could use this theory in practice, in order to obtain, for example, the supersymmetric standard model in the same spirit as one obtains the standard model of fundamental interactions from ordinary gauge theory. The notion of graded principal bundle has first appeared in [**1**], [**2**]: there, the notion of connection on such bundles has been also presented, but in a rather academic way, i.e. using jet bundles, bundles of connections and Atiyah's exact sequence, conveniently generalized in this context. This point of view turned out to be particularly useful for mathematics, as it inspired and affected more recent developments in connection theory on far more sophisticated types of supermanifolds, see [**3**].

Our analysis focuses on the investigation of the geometry of graded principal bundles, as well as on the reformulation of the notion of graded connection in terms of graded distributions and Lie superalgebra-valued graded differential forms. We believe that our approach makes the theory of graded connections more accessible and close to applications in supersymmetric Yang-Mills theory, providing thus the possibility to this elegant and economical theory to be applied on specific problems of supersymmetric physics.

Throughout the paper, we follow the original terminology of Kostant, [**15**] and we avoid the use of the term "supermanifold" or "Lie supergroup". This strict distinction between the terms "supermanifold" and "graded manifold" is justified if one wishes to avoid confusion with the DeWitt's or Roger's supermanifold theory, [**7**], [**13**], [**21**], [**22**], [**24**]. In this latter approach, one constructs the supermanifold following the pattern of ordinary differential geometry (using supercharts, superdifferentiability, etc.). In this way, the supermanifold is a topological space, and in particular, a Banach manifold, [**7**]. In Kostant-Leites approach, the supermanifold is a pair which consists of a usual differentiable manifold equipped with a sheaf satisfying certain properties.

Despite the fact that Kostant's graded manifolds seem to be more abstract, we have chosen to follow Kostant's formalism for several reasons. The most important is that in this formalism we avoid the complications of DeWitt's supergeometry coming from superdifferentiability of mappings. Furthermore, Kostant's graded manifold theory is closer to ordinary manifold theory, since it constitutes a special kind of sheaf theory over ordinary manifolds. Finally, the algebraic concepts entering in graded manifold theory often significantly simplify the calculus on graded manifolds.

The article is organized as follows. In section 2, we recall the fundamentals of graded manifold theory, [**15**], which will be necessary for the subsequent analysis, introducing at the same time some new concepts and tools, useful in computing pull-backs of graded differential forms. Next section deals with graded Lie theory. After a review of the basics of the theory, [**15**], [**1**], we show how one can obtain a right action of a graded Lie group from a left one (and vice versa); we also explain how a graded Lie group action gives rise to graded manifold morphisms and derivations. A graded analog of the adjoint action of a Lie group on itself is analyzed, and the parallelizability theorem for graded Lie groups is proved. In section 4 we investigate the relation between graded distributions and graded Lie group actions. The free action in graded geometry is the central notion of this section for which we provide two equivalent characterizations. The main result is that each free action of a graded Lie group induces a regular graded distribution. We also study quotient graded structures, completing the work of [**1**] on this subject.

The notion and the geometry of graded principal bundle are analyzed in sections 5 and 6. Here, we adopt a definition of this object slightly different from that in [**2**], completing in some sense, the definition of the last reference. We prove that several properties of ordinary principal bundles remain valid in the graded context, making of course the appropriate modifications and generalizations. For example, we prove that the orbits of the structure graded Lie group are identical to the fibres of the projection and that a graded principal bundle is globally trivial, if and only if admits a global graded section. As an application of the tools developed until now, we provide an alternative definition of the graded principal bundle, often very useful as we explain in section 6. After the introduction and study of the main properties of Lie superalgebra-valued graded differential forms in section 7, we are ready to introduce the notion of graded connection. Our definition is guided by the geometrical structure of the graded principal bundle: the sheaf of vertical derivations coincides with the graded distribution induced by the action of the structure graded Lie group. We discuss two equivalent definitions of the graded connection and we show how one can construct a graded connection locally. As a direct application, we establish the existence theorem for graded connections.

The graded curvature is the subject of section 9. We prove the structure equation and the Bianchi identity for the curvature of a graded connection. We show finally that this notion controls the involutivity of the horizontal graded distribution determined by a graded connection.

The previous analysis on the curvature shows also that the graded connection $\boldsymbol{\omega}$ (as well as its curvature $F^{\boldsymbol{\omega}}$) decompose as $\boldsymbol{\omega} = \boldsymbol{\omega}_0 + \boldsymbol{\omega}_1$, where $\boldsymbol{\omega}_0$ is an even graded differential 1-form with values in the even part of the Lie superalgebra $\mathfrak{g}$ of the structure group, while $\boldsymbol{\omega}_1$ is an odd graded differential 1-form with values in $\mathfrak{g}_1$, the odd part of $\mathfrak{g}$. Furthermore, the restriction of $\boldsymbol{\omega}_0$ to the underlying differentiable manifold (which is an ordinary principal bundle) gives rise to a usual connection and exactly this observation suggests to interpret $\boldsymbol{\omega}$, in physics terminology, as a supersymmetric gauge potential incorporating the usual gauge potential $\boldsymbol{\omega}_0$ and its supersymmetric partner $\boldsymbol{\omega}_1$.

Notational conventions. For an algebra A, A^* denotes its full dual and A°

its finite dual (in contrast to Kostant's conventions where A' denotes the full, and A^* the finite dual). If $\mathscr{A}$ is a sheaf of algebras over a differentiable manifold M, then $\mathbb{1}_{\mathscr{A}}$ denotes the unit of $\mathscr{A}(M)$ and $m_{\mathscr{A}}$ the algebra multiplication. Throughout this article the term "graded commutative" means "$\mathbf{Z}_2$-graded commutative", unless otherwise stated. If E is a $\mathbf{Z}_2$-graded vector space, then E_0 and E_1 stand for its even and odd subspaces: $E = E_0 \oplus E_1$. For an element $v \in E_i, i = 0, 1$, $|v| = i$ denotes the $\mathbf{Z}_2$-degree of v. Elements belonging only to E_0 or E_1 are called homogeneous (even or odd, respectively).

2. Graded manifold theory

In this section, we review the basic notions of graded manifold theory, [**15**]. We also introduce some new concepts following the pattern of ordinary differential geometry as well as tools that will be useful in the sequel. More precisely, the notions of vertical and projectable derivations are introduced; push-forward (and pull-back) of derivations under isomorphisms of graded manifolds is defined and some useful formulas for the computation of pull-backs of forms are established. We begin with the notion of the graded manifold, [**15**].

A ringed space $(M, \mathscr{A})$ is called a graded manifold of dimension (m, n) if M is a differentiable manifold of dimension m, $\mathscr{A}$ is a sheaf of graded commutative algebras, there exist a sheaf epimorphism ϱ: $\mathscr{A} \to C^\infty$ and for each open U, an open covering $\{V_i\}$ such that $\mathscr{A}(V_i)$ is isomorphic as a graded commutative algebra to $C^\infty(V_i) \otimes \Lambda\mathbf{R}^n$, in a manner compatible with the restriction morphisms of the sheaves involved. Here, C^∞ stands for the sheaf of differentiable functions on M, equipped with its trivial grading: $\big(C^\infty(U)\big)_0 = C^\infty(U)$ for each open subset $U \subset M$. We note $\dim(M, \mathscr{A}) = (m, n)$.

An open $U \subset M$, for which $\mathscr{A}(U) \cong C^\infty(U) \otimes \Lambda\mathbf{R}^n$, is called an $\mathscr{A}$-splitting neighborhood. A graded coordinate system on an $\mathscr{A}$-splitting neighborhood U is a collection $(x^i, s^j) = (x^1, \ldots, x^m, s^1, \ldots, s^n)$ of homogeneous elements of $\mathscr{A}(U)$ with $|x^i| = 0, |s^j| = 1$, such that $(\tilde{x}^1, \ldots, \tilde{x}^m)$ is an ordinary coordinate system on the open U, where $\tilde{x}^i = \varrho(x^i) \in C^\infty(U)$ and $(s^1, \ldots, s^n)$ are algebraically independent elements, that is, $\prod_{j=1}^n s^j \neq 0$. It can also be shown, [**15**], that if $\mathscr{A}^1(U)$ is

the set of nilpotent elements of $\mathscr{A}(U)$, then the sequence

$$0 \longrightarrow \mathscr{A}^1(U) \longrightarrow \mathscr{A}(U) \longrightarrow C^\infty(U) \longrightarrow 0 \tag{2.1}$$

is exact.

Given two graded manifolds $(X, \mathscr{A})$ and $(Y, \mathscr{B})$, one can form their product $(Z, \mathscr{C}) = (X, \mathscr{A}) \times (Y, \mathscr{B})$ which is also a graded manifold, [**23**], where $Z = X \times Y$ and the sheaf $\mathscr{C}$ is given by $\mathscr{C}(Z) = \mathscr{A}(X) \hat{\otimes}_\pi \mathscr{B}(Y)$; in the previous tensor product, π means the completion of $\mathscr{A}(X) \otimes \mathscr{B}(Y)$ with respect to Grothendieck's π-topology, [**11**], [**23**].

One can define a morphism between two graded manifolds as being a morphism of ringed spaces compatible with the sheaf epimorphism ϱ, [**23**], but often it is more convenient to do this in a more concise way: a morphism $\sigma\colon (M, \mathscr{A}) \to (N, \mathscr{B})$ between two graded manifolds is just a morphism $\sigma^*\colon \mathscr{B}(N) \to \mathscr{A}(M)$ of graded commutative algebras.

A very useful object in graded manifold theory is the finite dual $\mathscr{A}(M)^\circ$ of $\mathscr{A}(M)$ defined as

$$\mathscr{A}(M)^\circ = \{a \in \mathscr{A}(M)^* \mid a \text{ vanishes on an ideal of } \mathscr{A}(M) \text{ of finite codimension}\}.$$

Using general algebraic techniques (see for example [**19**]), one readily verifies that $\mathscr{A}(M)^\circ$ is a graded cocommutative coalgebra, the coproduct $\Delta^\circ_{\mathscr{A}}$ and counit $\epsilon^\circ_{\mathscr{A}}$ on $\mathscr{A}(M)^\circ$ being given by $\Delta^\circ_{\mathscr{A}} a(f \otimes g) = a(fg)$, $\epsilon^\circ_{\mathscr{A}}(a) = a(\mathbb{1}_{\mathscr{A}})$, $\forall a \in \mathscr{A}(M)^\circ$, $f, g \in \mathscr{A}(M)$. The set of group-like elements of this coalgebra contains only elements of the form δ_p for $p \in M$, where $\delta_p\colon \mathscr{A}(M) \to \mathbf{R}$ is defined by

$$\delta_p(f) = \varrho(f)(p) = \tilde{f}(p). \tag{2.2}$$

Furthermore, if $\sigma\colon (M, \mathscr{A}) \to (N, \mathscr{B})$ is a morphism of graded manifolds, then the element $\sigma_* a \in \mathscr{B}(N)^*$ defined for $a \in \mathscr{A}(M)^\circ$ by

$$\sigma_* a(g) = a(\sigma^* g), \forall g \in \mathscr{B}(N) \tag{2.3}$$

vanishes on an ideal of finite codimension. We thus obtain a morphism of graded coalgebras $\sigma_*\colon \mathscr{A}(M)^\circ \to \mathscr{B}(N)^\circ$ which respects the group-like elements and induces a differentiable map $\sigma_*|_M\colon M \to N$.

Another very important property of the graded coalgebra $\mathscr{A}(M)^\circ$ is that the set of its primitive elements with respect to the group-like element δ_p, i.e. elements $v \in \mathscr{A}(M)^\circ$ for which $\Delta^\circ_{\mathscr{A}}(v) = v \otimes \delta_p + \delta_p \otimes v$, is equal to the set of derivations at p on $\mathscr{A}(M)$, that is, the set of elements $v \in \mathscr{A}(M)^*$ for which

$$v(fg) = (vf)(\delta_p g) + (-1)^{|v||f|}(\delta_p f)(vg), \forall f, g \in \mathscr{A}(M) \tag{2.4}$$

if v and f are homogeneous. This set is a subspace of $\mathscr{A}(M)^\circ$ which we call tangent space of $(M, \mathscr{A})$ at $p \in M$ and we note it by $T_p(M, \mathscr{A})$. One easily verifies that the morphisms of graded manifolds preserve the subspaces of primitive elements: if $v \in T_p(M, \mathscr{A})$, then $\sigma_* v \in T_q(N, \mathscr{B})$ where $q = \sigma_*|_M(p)$. Hence, we have a well-defined notion of the tangent (or the differential) of the morphism σ at any point $p \in M$ and we adopt the notation

$$T_p\sigma \colon T_p(M, \mathscr{A}) \to T_q(N, \mathscr{B}), \quad T_p\sigma(v) = \sigma_* v. \tag{2.5}$$

In this context, the set $\mathfrak{Der}\mathscr{A}(U)$ of derivations of $\mathscr{A}(U)$ plays the rôle of graded vector fields on $(U, \mathscr{A}|_U), U \subset M$ open. The difference with the ordinary differential geometry is that we cannot evaluate directly a derivation $\xi \in \mathfrak{Der}\mathscr{A}(U)$ at a point $p \in U$ in order to obtain a tangent vector belonging to $T_p(M, \mathscr{A})$. Instead, we may associate to each $\xi \in \mathfrak{Der}\mathscr{A}(U)$ a tangent vector $\tilde{\xi}_p \in T_p(M, \mathscr{A}), \forall p \in U$ in the following way: for each $f \in \mathscr{A}(U)$, we define

$$\tilde{\xi}_p(f) = \delta_p(\xi f). \tag{2.6}$$

If U is an open subset of M and $(m, n) = \dim(M, \mathscr{A})$, the set of derivations $\mathfrak{Der}\mathscr{A}(U)$ is a free left $\mathscr{A}(U)$-module of dimension (m, n). If U is an $\mathscr{A}$-splitting neighborhood, (x^i, s^j) a graded coordinate system on U and $\xi \in \mathfrak{Der}\mathscr{A}(U)$, then there exist elements $\xi^i, \xi^j \in \mathscr{A}(U)$ such that

$$\xi = \sum_{i=1}^{m} \xi^i \frac{\partial}{\partial x^i} + \sum_{j=1}^{n} \xi^j \frac{\partial}{\partial s^j},$$

where the derivations $\partial/\partial x^i$ and $\partial/\partial s^j$ are defined by

$$\frac{\partial}{\partial s^k}(s^\ell) = \delta^\ell_k \mathbb{1}_U, \;\; \frac{\partial}{\partial s^k}(x^i) = 0, \;\; \frac{\partial}{\partial x^i}(s^k) = 0, \;\; \frac{\partial}{\partial x^i}(x^j) = \delta^j_i \mathbb{1}_U, \tag{2.7}$$

$\mathbb{1}_U$ being the unit of $\mathscr{A}(U)$. Clearly, this decomposition is not valid in general for derivations belonging to $\mathfrak{Der}\mathscr{A}(M)$. Therefore, we give the following definition:

2.1 Definition. *A graded manifold $(M, \mathscr{A})$ is called parallelizable if the set of derivations $\mathfrak{Der}\mathscr{A}(M)$ admits a global basis on $\mathscr{A}(M)$ consisting of m even and n odd derivations.*

The difficulty one encounters in ordinary manifold theory to push-forward (or to pull-back) a vector field by means of a differentiable mapping is also present in the context of graded manifolds. As in the case of ordinary manifolds, this is possible only if we use isomorphisms of the graded manifold structure. More precisely:

2.2 Definition. *Let $\sigma\colon (M, \mathscr{A}) \to (N, \mathscr{B})$ be an isomorphism between the graded manifolds $(M, \mathscr{A})$ and $(N, \mathscr{B})$ and let $U \subset M$ be an open subset such that $U = \sigma_*^{-1}(V), V \subset N$ open. If $\xi \in \mathfrak{Der}\mathscr{A}(U)$, then we define the push-forward $\sigma_*\xi \in \mathfrak{Der}\mathscr{B}(V)$ as $\sigma_*\xi = (\sigma^*)^{-1} \circ \xi \circ \sigma^*$. For the pull-back of $\eta \in \mathfrak{Der}\mathscr{B}(V)$, we define $\sigma^*\eta = (\sigma^{-1})_*\eta \in \mathfrak{Der}\mathscr{A}(U)$.*

In many situations it happens that some derivations may be related through a morphism (which is not necessarily an isomorphism) in a manner similar to those of pull-back. We give the following definition:

2.3 Definition. *Let $\sigma\colon (M, \mathscr{A}) \to (N, \mathscr{B})$ be a morphism of graded manifolds. We call two derivations $\xi \in \mathfrak{Der}\mathscr{A}(M)$ et $\eta \in \mathfrak{Der}\mathscr{B}(N)$ σ-related if for each $f \in \mathscr{B}(N)$ we have $\sigma^*(\eta f) = \xi(\sigma^* f)$. Especially, if we fix ξ and σ is an epimorphism, the derivation η, if it exists, is unique; in such a case we note $\eta = \sigma_*\xi$, we call ξ a σ_*-projectable derivation and $\sigma_*\xi$ its projection by means of σ. ξ will be called vertical derivation if it is σ_*-projectable and $\sigma_*\xi = 0$.*

It is easy to verify that the set $\mathscr{Pro}(\sigma_*, \mathscr{A})(M)$ of σ_*-projectable derivations is a left $\mathscr{B}(N)$-module and that the set $\mathscr{Ver}(\sigma_*, \mathscr{A})(M)$ of vertical derivations is a left $\mathscr{A}(M)$-module. Indeed, for $g \in \mathscr{B}(N)$ and $\xi \in \mathscr{Pro}(\sigma_*, \mathscr{A})(M)$ let us define $g \cdot \xi \in \mathfrak{Der}\mathscr{A}(M)$ as $g \cdot \xi = (\sigma^* g)\xi$. Then for $f \in \mathscr{B}(N)$ we find: $(g \cdot \xi)(\sigma^* f) = (\sigma^* g)\xi(\sigma^* f) = \sigma^* g \cdot \sigma^*[\sigma_*\xi(f)] = \sigma^*[g(\sigma_*\xi)f]$, which proves that

$g \cdot \xi \in \mathscr{P}\!ro(\sigma_*, \mathscr{A})(M)$ as well. We proceed analogously for $\mathscr{V}\!er(\sigma_*, \mathscr{A})(M)$. The corresponding sheaves are denoted by $\mathscr{P}\!ro(\sigma_*, \mathscr{A}), \mathscr{V}\!er(\sigma_*, \mathscr{A})$.

The final part of this section is devoted to a short review of the basic properties of graded differential forms from [**15**]. We also develop some useful tools for computing pull-backs of forms using the previously mentioned concepts of push-forward and σ-related derivations.

Let then $(M, \mathscr{A})$ be a graded manifold of dimension (m, n). For an open $U \subset M$ we consider the tensor algebra $\mathfrak{T}(U)$ of $\mathfrak{Der}\mathscr{A}(U)$ with respect to its $\mathscr{A}(U)$-module structure and the ideal $\mathfrak{J}(U)$ of $\mathfrak{T}(U)$ generated by homogeneous elements of the form $\xi \otimes \eta + (-1)^{|\xi||\eta|}\eta \otimes \xi$, for $\xi, \eta \in \mathfrak{Der}\mathscr{A}(U)$. Let also $\mathfrak{T}^r(U) \cap \mathfrak{J}(U) = \mathfrak{J}^r(U)$. We call set of graded differential r-forms ($r \geq 1$) on $U \subset M$ the set $\Omega^r(U, \mathscr{A})$ of elements belonging to $\mathrm{Hom}_{\mathscr{A}(U)}(\mathfrak{T}^r(U), \mathscr{A}(U))$ which vanish on $\mathfrak{J}^r(U)$. For $r = 0$ we define $\Omega^0(U, \mathscr{A}) = \mathscr{A}(U)$ and we note the direct sum $\bigoplus_{p=0}^{\infty} \Omega^r(U, \mathscr{A})$ by $\Omega(U, \mathscr{A})$.

If $\alpha \in \Omega^r(U, \mathscr{A})$ and $\xi_1, \ldots, \xi_r \in \mathfrak{Der}\mathscr{A}(U)$, then we denote the evaluation of α on the ξ's by $(\xi_1 \otimes \ldots \otimes \xi_r|\alpha)$, or simply $(\xi_1, \ldots, \xi_r|\alpha)$.

Clearly, the elements de $\Omega(U, \mathscr{A})$ have a $(\mathbf{Z} \oplus \mathbf{Z}_2)$-bidegree and further we may define an algebra structure on $\Omega(U, \mathscr{A})$ which thus becomes a bigraded commutative algebra over $\mathscr{A}(U)$, [**15**]. Here, we mention only the bigraded commutativity relation for this structure: if $\alpha \in \Omega^{i_\alpha}(U, \mathscr{A})_{j_\alpha}, \beta \in \Omega^{i_\beta}(U, \mathscr{A})_{j_\beta}$, then $\alpha\beta \in \Omega^{i_\alpha + i_\beta}(U, \mathscr{A})_{j_\alpha + j_\beta}$ and $\alpha\beta = (-1)^{i_\alpha i_\beta + j_\alpha j_\beta}\beta\alpha$.

Next consider the linear map d: $\Omega^0(U, \mathscr{A}) \to \Omega^1(U, \mathscr{A})$ defined by

$$(\xi|dg) = \xi(g), \;\; \xi \in \mathfrak{Der}\mathscr{A}(U), g \in \mathscr{A}(U). \tag{2.8}$$

In a graded coordinate system $(x^i, s^j), i = 1, \ldots m, j = 1, \ldots n$ on U, we take

$$dg = \sum_{i=1}^{m} dx^i \frac{\partial g}{\partial x^i} + \sum_{j=1}^{n} ds^j \frac{\partial g}{\partial s^j}. \tag{2.9}$$

One can extend this linear map to a derivation d: $\Omega(U, \mathscr{A}) \to \Omega(U, \mathscr{A})$ of bidegree (1,0) such that $d^2 = 0$ and $d|_{\Omega^0(U,\mathscr{A})}$ gives equation (2.9). We will call d exterior differential on graded differential forms.

Interior products and Lie derivatives with respect to elements of $\mathfrak{Der}\mathscr{A}(U)$ also make sense in the graded setting. Indeed, if $\alpha \in \Omega^{r+1}(U, \mathscr{A})$, then for $\xi, \xi_1, \dots, \xi_r$ homogeneous elements of $\mathfrak{Der}\mathscr{A}(U)$ we define

$$(\xi_1, \dots, \xi_r | \boldsymbol{i}(\xi)\alpha) = (-1)^{|\xi| \sum_{i=1}^{r} |\xi_i|} (\xi, \xi_1, \dots, \xi_r | \alpha) \tag{2.10}$$

and we thus obtain a linear map $\boldsymbol{i}(\xi)\colon \Omega(U, \mathscr{A}) \to \Omega(U, \mathscr{A})$ of bidegree $(-1, |\xi|)$. This is the interior product with respect to ξ.

Lie derivatives are defined as usual by means of Cartan's algebraic formula: $\boldsymbol{L}_\xi = d \circ \boldsymbol{i}(\xi) + \boldsymbol{i}(\xi) \circ d$, thus $\boldsymbol{L}_\xi$ has bidegree $(0, |\xi|)$. Furthermore, it can be proved that the morphism of graded commutative algebras $\sigma^*\colon \mathscr{B}(W) \to \mathscr{A}(U), U = \sigma_*^{-1}(W) \subset M$ coming from a morphism $\sigma\colon (M, \mathscr{A}) \to (N, \mathscr{B})$ of graded manifolds, can be extended to a unique morphism of bigraded commutative algebras $\sigma^*\colon \Omega(W, \mathscr{B}) \to \Omega(U, \mathscr{A})$, which commutes with the exterior differential.

As for the case of derivations, see relation (2.6), one can define for each graded differential form $\alpha \in \Omega^r(U, \mathscr{A})$ a multilinear form $\tilde{\alpha}_p$ (with real values) on the tangent space $T_p(M, \mathscr{A})$ for each $p \in U$. It suffices to set

$$\big((\tilde{\xi}_1)_p, \dots, (\tilde{\xi}_r)_p | \tilde{\alpha}_p\big) = \delta_p(\xi_1, \dots, \xi_r | \alpha). \tag{2.11}$$

The set of forms on U obtained in this way is denoted by $\Omega^r_{\mathscr{A}}(U)$.

We establish now a general method for the calculation of pull-backs of graded differential forms.

2.4 Proposition. *Let $\sigma\colon (M, \mathscr{A}) \to (N, \mathscr{B})$ be a morphism of graded manifolds, $W \subset N$ and $U = \sigma_*^{-1}(W) \subset M$. Then, if $\alpha \in \Omega^r(W, \mathscr{B})$ and $\xi_i \in \mathfrak{Der}\mathscr{A}(U)$ and $\eta_i \in \mathfrak{Der}\mathscr{B}(W)$ are σ-related for $i = 1, \dots, r$, we have:*

$$(\xi_1, \dots, \xi_r | \sigma^*\alpha) = \sigma^*(\eta_1, \dots, \eta_r | \alpha). \tag{2.12}$$

In particular, when σ is an isomorphism, the previous relation holds for each $\xi_i \in \mathfrak{Der}\mathscr{A}(U)$, setting $\eta_i = \sigma_*\xi_i$.

Proof. Using the fact that σ^* is an isomorphism of bigraded commutative algebras and that $\Omega(W, \mathscr{B})$ is the exterior algebra of $\Omega^1(W, \mathscr{B})$, it is sufficient to

prove this formula for $\alpha = df, f \in \mathscr{B}(W)$. If $\xi \in \mathfrak{Der}\mathscr{A}(U)$ and $\eta \in \mathfrak{Der}\mathscr{B}(W)$ are σ-related, then $(\xi|\sigma^*\alpha) = (\xi|d(\sigma^* f)) = \xi(\sigma^* f) = \sigma^*(\eta f) = \sigma^*(\eta|df) = \sigma^*(\eta|\alpha)$. In particular, when σ is an isomorphism, each ξ is σ-related to $\eta = \sigma_*\xi$. ■

One easily proves that pull-backs on forms belonging to $\Omega^r_{\mathscr{B}}(W)$ can be expressed using the familiar formulas of ordinary differential geometry but taking as tangent of the morphism σ, the linear mapping defined through relations (2.3), (2.4).

3. Elements of graded Lie theory

As we will see later in detail, graded Lie theory plays a central rôle in the geometry of graded principal bundles just as ordinary theory of Lie groups does in differentiable principal bundles. This section deals with the notion and elementary properties of graded Lie groups; more information can be found in [**15**]. Furthermore, we prove some facts about the theory of actions of graded Lie groups, which, to the best of our knowledge, have not ever appeared in the literature. We first give the definition of a graded Lie group, [**1**], [**15**].

3.1 Definition. *A graded Lie group $(G, \mathscr{A})$ is a graded manifold such that G is an ordinary Lie group, the algebra $\mathscr{A}(G)$ is equipped with the structure of a graded Hopf algebra with antipode and furthermore, the algebra epimorphism $\varrho\colon \mathscr{A}(G) \to C^\infty(G)$ is a morphism of graded Hopf algebras.*

3.2 Remark. *In the graded Hopf algebra structure of the previous definition, all tensor products are completions of the usual ones with respect to Grothendieck's π-topology,* [**1**], [**23**].

We denote by $\Delta_{\mathscr{A}}$, $\epsilon_{\mathscr{A}}$, $s_{\mathscr{A}}$ the coproduct, counit and antipode of $\mathscr{A}(G)$, respectively. It is possible to prove that the finite dual $\mathscr{A}(G)^\circ$ inherits also a graded Hopf algebra structure from $\mathscr{A}(G)$. The algebra multiplication on $\mathscr{A}(G)^\circ$ is given by the convolution product:

$$(a \odot b) = (a \otimes b) \circ \Delta_{\mathscr{A}}, \ \ \forall a, b \in \mathscr{A}(G)^\circ, \tag{3.1}$$

and the unit of $\mathscr{A}(G)^\circ$ with respect to $\odot$ is the counit of $\mathscr{A}(G)$. One proves easily that the set of primitive elements of $\mathscr{A}(G)^\circ$ with respect to δ_e (e is the identity of G), that is, the tangent space $T_e(G,\mathscr{A})$, is a graded Lie algebra, the bracket being given by $[u,v] = u \odot v - (-1)^{|u||v|} v \odot u$, for homogeneous elements u, v. We call $T_e(G,\mathscr{A})$ Lie superalgebra of $(G.\mathscr{A})$ and denote it by $\mathfrak{g}$. Clearly, $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$, where $\mathfrak{g}_0 = T_e G$.

Using also the fact that $\varrho\colon \mathscr{A}(G) \to C^\infty(G)$ is a morphism of graded Hopf algebras one readily verifies that the convolution product $\odot$ is compatible with the group structure of G in the sense that $\delta_g \odot \delta_h = \delta_{gh}$.

A very important property of the finite dual $\mathscr{A}(G)^\circ$ is given by the following, [**15**].

3.3 Theorem. *For each graded Lie group $(G,\mathscr{A})$, the finite dual $\mathscr{A}(G)^\circ$ has the structure of a Lie-Hopf algebra. In fact, $\mathscr{A}(G)^\circ = \mathbf{R}(G) \circledast E(\mathfrak{g})$, where $\mathbf{R}(G)$ is the group algebra of G, $E(\mathfrak{g})$ is the universal enveloping algebra of $\mathfrak{g}$ and $\circledast$ is the smash product of $\mathbf{R}(G)$ and $E(\mathfrak{g})$ with respect to the adjoint representation of G on the superalgebra $\mathfrak{g}$.*

For the adjoint representation of G on $\mathfrak{g}$, see also below in this section.

There exist, in this setting, analogs of left and right translations on a Lie group.

3.4 Definition. *Let $(G,\mathscr{A})$ be a graded Lie group and $a \in \mathscr{A}(G)^\circ$. Set $r_a = (id \otimes a) \circ \Delta_{\mathscr{A}}$ and $\ell_a = (a \otimes id) \circ \Delta_{\mathscr{A}}$. We call the endomorphisms r_a and ℓ_a of $\mathscr{A}(G)$ right and left translations respectively on $(G,\mathscr{A})$.*

In order to justify this terminology, we first discuss some important properties of these endomorphisms.

3.5 Proposition.

1. $r_{\epsilon_{\mathscr{A}}} = \ell_{\epsilon_{\mathscr{A}}} = id$
2. $r_{a \odot b} = r_a \circ r_b$, $\ell_{a \odot b} = (-1)^{|a||b|} \ell_b \circ \ell_a$
3. $r_b \circ \ell_a = (-1)^{|a||b|} \ell_a \circ r_b, \forall a, b \in \mathscr{A}(G)^\circ$
4. *If $a \in \mathscr{A}(G)^\circ$ is group-like, then r_a and ℓ_a are graded algebra isomorphisms.*

We postpone the proof until the study of actions of graded Lie groups (see below in this section). Then, it will be clear that the previous proposition is immediate using the general techniques of actions (note that Proposition 3.5 has already appeared in [**1**] without proof).

Part (4) of this proposition tells us that if a is group-like $a = \delta_g$, $g \in G$, then there exist morphisms of graded manifolds R_g: $(G, \mathscr{A}) \to (G, \mathscr{A})$, L_g: $(G, \mathscr{A}) \to (G, \mathscr{A})$ such that $R_g^* = r_a$ and $L_g^* = \ell_a$. It is interesting to calculate the coalgebra morphisms R_{g*}: $\mathscr{A}(G)^\circ \to \mathscr{A}(G)^\circ$ and L_{g*}: $\mathscr{A}(G)^\circ \to \mathscr{A}(G)^\circ$. Consider for example R_{g*}. If $b \in \mathscr{A}(G)^\circ$ and $f \in \mathscr{A}(G)$, we find:

$$R_{g*}b(f) = b(r_a f) = b\big(\sum_i (-1)^{|a||I^i f|}(I^i f)a(J^i f)\big) = \sum_i (b \otimes a)(I^i f \otimes J^i f),$$

where we have set $\Delta_{\mathscr{A}} f = \sum_i I^i f \otimes J^i f$. Hence, $R_{g*}b = b \odot a$ and similarly $L_{g*}b = a \odot b$. This means that r_a and ℓ_a correspond to right and left translations, as one can see at the coalgebra level.

Next, we introduce the graded analog of actions (see also [**1**]). Let then $(G, \mathscr{A})$ be a graded Lie group and $(Y, \mathscr{B})$ a graded manifold. We give the following definition.

3.6 Definition. *We say that the graded Lie group $(G, \mathscr{A})$ acts on the graded manifold $(Y, \mathscr{B})$ to the right if there exists a morphism Φ: $(Y, \mathscr{B}) \times (G, \mathscr{A}) \to (Y, \mathscr{B})$ of graded manifolds such that the corresponding morphism of graded commutative algebras Φ^*: $\mathscr{B}(Y) \to \mathscr{B}(Y) \hat{\otimes}_\pi \mathscr{A}(G)$ defines a structure of right $\mathscr{A}(G)$-comodule on $\mathscr{B}(Y)$. Using the notion of left comodule, we may define the left action of $(G, \mathscr{A})$ on $(Y, \mathscr{B})$.*

More explicitly, if Φ is a right and Ψ is a left action, then the morphisms Φ^*, Ψ^* satisfy the following properties:

$$(id \otimes \Delta_{\mathscr{A}}) \circ \Phi^* = (\Phi^* \otimes id) \circ \Phi^*, \quad (id \otimes \epsilon_{\mathscr{A}}) \circ \Phi^* = id, \tag{3.2}$$

$$(\Delta_{\mathscr{A}} \otimes id) \circ \Psi^* = (id \otimes \Psi^*) \circ \Psi^*, \quad (\epsilon_{\mathscr{A}} \otimes id) \circ \Psi^* = id. \tag{3.3}$$

Let now Ψ^r: $(Y, \mathscr{B}) \times (G, \mathscr{A}) \to (Y, \mathscr{B})$ be the morphism of graded manifolds defined by

$$\Psi^{r*} = (id \otimes s_{\mathscr{A}}) \circ T \circ \Psi^*, \tag{3.4}$$

where T is the twist morphism, $T(a \otimes b) = (-1)^{|a||b|} b \otimes a$.

3.7 Lemma. *The morphism* Ψ^r *defined by* (3.4) *is a right action of* $(G, \mathscr{A})$ *on* $(Y, \mathscr{B})$.

Proof. It suffices to prove that relations (3.2) are valid for the morphism Ψ^{r*}. We take: $(id \otimes \Delta_{\mathscr{A}}) \circ \Psi^{r*} = \big(id \otimes (s_{\mathscr{A}} \otimes s_{\mathscr{A}}) \circ T \circ \Delta_{\mathscr{A}}\big) \circ T \circ \Psi^* = \big(id \otimes (s_{\mathscr{A}} \otimes s_{\mathscr{A}}) \circ T\big) \circ T \circ (\Delta_{\mathscr{A}} \otimes id) \circ \Psi^* = \big(id \otimes (s_{\mathscr{A}} \otimes s_{\mathscr{A}}) \circ T\big) \circ T \circ (id \otimes \Psi^*) \circ \Psi^* = (id \otimes s_{\mathscr{A}} \otimes s_{\mathscr{A}}) \circ (T \circ \Psi^* \otimes id) \circ T \circ \Psi^* = (\Psi^{r*} \otimes id) \circ \Psi^{r*}$. On the other hand,

$$(id \otimes \epsilon_{\mathscr{A}}) \circ \Psi^{r*} = (id \otimes \epsilon_{\mathscr{A}} \circ s_{\mathscr{A}}) \circ T \circ \Psi^* = (id \otimes \epsilon_{\mathscr{A}}) \circ T \circ \Psi^* = id,$$

which completes the proof. ■

Similarly, for a right action $\Phi\colon (Y, \mathscr{B}) \times (G, \mathscr{A}) \to (Y, \mathscr{B})$ one can define in a canonical way, a left action $\Phi^\ell\colon (G, \mathscr{A}) \times (Y, \mathscr{B}) \to (Y, \mathscr{B})$ as $\Phi^{\ell *} = (s_{\mathscr{A}} \otimes id) \circ T \circ \Phi^*$. Then, the restriction $\Phi_*|_{Y \times G}\colon Y \times G \to Y$ defines a right action of G on the manifold Y and furthermore, for the canonically associated left action Φ^ℓ, the restriction $\Phi^\ell_*|_{G \times Y}\colon G \times Y \to Y$ is a left action given by $\Phi^\ell_*|_{G \times Y}(g, y) = \Phi_*|_{Y \times G}(y, g^{-1})$, as one expects. We have analogous facts for the left action Ψ. Observe here that the possibility to define $\Phi^{\ell *}$ and Ψ^{r*} as morphisms of graded commutative algebras depends crucially on the fact that the antipode $s_{\mathscr{A}}\colon \mathscr{A}(G) \to \mathscr{A}(G)$ is a morphism of graded commutative algebras.

3.8 Remark. *The right action* $\Phi^{\ell r}$ *canonically associated to* Φ^ℓ *equals to* Φ*:* $\Phi^{\ell r *} = (id \otimes s_{\mathscr{A}}) \circ T \circ (s_{\mathscr{A}} \otimes id) \circ T \circ \Phi^* = (id \otimes s_{\mathscr{A}}) \circ T^2 \circ (id \otimes s_{\mathscr{A}}) \circ \Phi^* = \Phi^*$*, since* $T^2 = id, s^2_{\mathscr{A}} = id$.

For a right action Φ, one may introduce for each $a \in \mathscr{A}(G)^\circ$ and $b \in \mathscr{B}(Y)^\circ$, two linear maps $(\Phi^*)_a\colon \mathscr{B}(Y) \to \mathscr{B}(Y)$ and $(\Phi^*)_b\colon \mathscr{B}(Y) \to \mathscr{A}(G)$ as follows:

$$(\Phi^*)_a = (id \otimes a) \circ \Phi^* \quad \text{and} \quad (\Phi^*)_b = (b \otimes id) \circ \Phi^*. \tag{3.5}$$

Similarly, for a left action Ψ one defines

$$(\Psi^*)_a = (a \otimes id) \circ \Psi^* \quad \text{and} \quad (\Psi^*)_b = (id \otimes b) \circ \Psi^*. \tag{3.6}$$

The following theorem clarifies the rôle of these maps.

3.9 Theorem.

1. $(\Phi^*)_{\epsilon_{\mathscr{A}}} = (\Psi^*)_{\epsilon_{\mathscr{A}}} = id$
2. $(\Phi^*)_{a_1 \odot a_2} = (\Phi^*)_{a_1} \circ (\Phi^*)_{a_2}$, $(\Psi^*)_{a_1 \odot a_2} = (-1)^{|a_1||a_2|}(\Psi^*)_{a_2} \circ (\Psi^*)_{a_1}$
3. $(\Phi^*)_b \circ (\Phi^*)_a = (-1)^{|a||b|} r_a \circ (\Phi^*)_b$, $(\Psi^*)_b \circ (\Psi^*)_a = (-1)^{|a||b|} \ell_a \circ (\Psi^*)_b$
4. *If a, b are group-like elements, then $(\Phi^*)_a$ is an isomorphism and $(\Phi^*)_b$ is a morphism of graded commutative algebras. In particular, if $a = \delta_g, b = \delta_y$, then we write the corresponding morphisms of graded manifolds as Φ_g: $(Y, \mathscr{B}) \to (Y, \mathscr{B})$ and Φ_y: $(G, \mathscr{A}) \to (Y, \mathscr{B})$, so $\Phi_g^* = (\Phi^*)_{\delta_g}$ and $\Phi_y^* = (\Phi^*)_{\delta_y}$. Similarly for $(\Psi^*)_a, (\Psi^*)_b$.*
5. *If a is a primitive element with respect to δ_e, then $(\Phi^*)_a, (\Psi^*)_a \in \mathfrak{Der}\mathscr{B}(Y)$. We call these derivations the induced (by the action and the element a) derivations on $\mathscr{B}(Y)$.*

<u>*Proof.*</u>

(1) Evident, by the defining properties of the left and right action.

(2) We prove this property only for the right action Φ; one proceeds in a similar way for the left action Ψ. We have:

$$
\begin{aligned}
(\Phi^*)_{a_1 \odot a_2} &= (id \otimes (a_1 \odot a_2)) \circ \Phi^* = (id \otimes a_1 \otimes a_2) \circ (id \otimes \Delta_{\mathscr{A}}) \circ \Phi^* \\
&= (id \otimes a_1 \otimes a_2) \circ (\Phi^* \otimes id) \circ \Phi^* = (id \otimes a_1) \circ \Phi^* \circ (id \otimes a_2) \circ \Phi^* \\
&= (\Phi^*)_{a_1} \circ (\Phi^*)_{a_2}.
\end{aligned}
$$

(3) Again, we give the proof only for the right action.

$$
\begin{aligned}
(\Phi^*)_b \circ (\Phi^*)_a &= (b \otimes id) \circ \Phi^* \circ (id \otimes a) \circ \Phi^* \\
&= (-1)^{|a||b|}(id \otimes a)(b \otimes id) \circ (id \otimes \Delta_{\mathscr{A}}) \circ \Phi^* \\
&= (-1)^{|a||b|}(id \otimes a) \circ \Delta_{\mathscr{A}} \circ (b \otimes id) \circ \Phi^* = (-1)^{|a||b|} r_a \circ (\Phi^*)_b.
\end{aligned}
$$

(4) The fact that these maps are morphisms is evident because they are compositions of morphisms when a, b are group-like. Furthermore, $(\Phi^*)_a$ and $(\Psi^*)_a$ are isomorphisms because their inverses exist, as one can check from parts (1) and (2).

(5) A derivation ξ on the graded commutative algebra $\mathscr{B}(Y)$ has the property $\xi(fg) = \xi(f)g + (-1)^{|\xi||f|} f\xi(g)$ for each homogeneous element $f \in \mathscr{B}(Y)$. This can be restated as follows: $\xi \circ m_{\mathscr{B}} = m_{\mathscr{B}} \circ (\xi \otimes id + id \otimes \xi)$. We call the endomorphism ξ primitive, so the set of derivations coincides with the set of primitive elements. Using this terminology, we must prove that $(\Phi^*)_a$ and $(\Psi^*)_a$ are primitive when a is primitive with respect to δ_e. Consider for example $\xi = (\Phi^*)_a$. We have:

$$\begin{aligned}
\xi \circ m_{\mathscr{B}} &= (m_{\mathscr{B}} \otimes a \circ m_{\mathscr{A}}) \circ (id \otimes T \otimes id) \circ (\Phi^* \otimes \Phi^*) \\
&= (m_{\mathscr{B}} \otimes (a \otimes \delta_e + \delta_e \otimes a)) \circ (id \otimes T \otimes id) \circ (\Phi^* \otimes \Phi^*) \\
&= m_{\mathscr{B}} \circ [(id \otimes a) \circ \Phi^* \otimes id] + m_{\mathscr{B}} \circ [id \otimes (id \otimes a) \circ \Phi^*] \\
&= m_{\mathscr{B}} \circ (\xi \otimes id + id \otimes \xi).
\end{aligned}$$

One proceeds similarly for $(\Psi^*)_a$. We note finally that if a is homogeneous, then $|(\Phi^*)_a| = |(\Psi^*)_a| = |a|$. ■

3.10 Corollary. *Proposition 3.5.*

Proof. Since the coproduct $\Delta_{\mathscr{A}}$ on the Hopf algebra $\mathscr{A}(G)$ has the properties $(id \otimes \Delta_{\mathscr{A}}) \circ \Delta_{\mathscr{A}} = (\Delta_{\mathscr{A}} \otimes id) \circ \Delta_{\mathscr{A}}$ and $(id \otimes \epsilon_{\mathscr{A}}) \circ \Delta_{\mathscr{A}} = (\epsilon_{\mathscr{A}} \otimes id) \circ \Delta_{\mathscr{A}} = id$, it defines left and right actions L and R respectively of $(G, \mathscr{A})$ on itself. Choosing thus $(Y, \mathscr{B}) = (G, \mathscr{A})$ in the previous theorem, we may write $(L^*)_a = \ell_a$, $(R^*)_a = r_a$; Proposition 3.5 is then immediate. ■

Next, let $(G, \mathscr{A})$ be a graded Lie group and ℓ: $\mathscr{A}(G) \to \mathscr{A}(G) \hat{\otimes}_\pi \mathscr{A}(G)$ the linear map defined by

$$\ell = [m_{\mathscr{A}} \circ (id \otimes s_{\mathscr{A}}) \otimes id] \circ (id \otimes T) \circ (id \otimes \Delta_{\mathscr{A}}) \circ \Delta_{\mathscr{A}}. \tag{3.7}$$

3.11 Proposition. *The linear map ℓ is a morphism of graded commutative algebras defining thus a morphism of graded manifolds which we denote by AD: $(G, \mathscr{A}) \times (G, \mathscr{A}) \to (G, \mathscr{A})$. Furthermore, AD is a left action of $(G, \mathscr{A})$ on itself.*

Proof. ℓ is a morphism of graded algebras as composition of morphisms; so we can write $\ell = AD^*$ for a morphism of graded manifolds AD: $(G, \mathscr{A}) \times (G, \mathscr{A}) \to$

$(G, \mathscr{A})$. We now check relations (3.3) for AD^*. For the first one, the following identity is the key of the proof: $(id \otimes id \otimes id \otimes \Delta_{\mathscr{A}}) \circ (id \otimes id \otimes \Delta_{\mathscr{A}}) \circ (\Delta_{\mathscr{A}} \otimes id) \circ \Delta_{\mathscr{A}} = (id \otimes id \otimes \Delta_{\mathscr{A}} \otimes id) \circ (id \otimes \Delta_{\mathscr{A}} \otimes id) \circ (id \otimes \Delta_{\mathscr{A}}) \circ \Delta_{\mathscr{A}}$. Indeed, applying the two members of the previous identity on the same $f \in \mathscr{A}(G)$, we find after a long and cumbersome calculation the first of (3.3). For the second of (3.3), we proceed as follows:

$$
\begin{aligned}
(\epsilon_{\mathscr{A}} \otimes id) \circ \ell &= (\epsilon_{\mathscr{A}} \circ m_{\mathscr{A}} \otimes id) \circ (id \otimes s_{\mathscr{A}} \otimes id) \circ (id \otimes T) \circ (id \otimes \Delta_{\mathscr{A}}) \circ \Delta_{\mathscr{A}} \\
&= (\epsilon_{\mathscr{A}} \otimes \epsilon_{\mathscr{A}} \circ s_{\mathscr{A}} \otimes id) \circ (id \otimes T) \circ (id \otimes \Delta_{\mathscr{A}}) \circ \Delta_{\mathscr{A}} \\
&= (\epsilon_{\mathscr{A}} \otimes id \otimes \epsilon_{\mathscr{A}}) \circ (id \otimes \Delta_{\mathscr{A}}) \circ \Delta_{\mathscr{A}} = (\epsilon_{\mathscr{A}} \otimes id) \circ \Delta_{\mathscr{A}} = id,
\end{aligned}
$$

which completes the proof. ■

We call the action AD adjoint action of $(G, \mathscr{A})$ on itself. As in ordinary Lie theory, the adjoint action respects the primitive elements with respect to δ_e in the sense of the following proposition.

3.12 Proposition. *Let $AD_{*a}\colon \mathscr{A}(G)^\circ \to \mathscr{A}(G)^\circ, a \in \mathscr{A}(G)^\circ$ be defined as $AD_{*a}(b) = AD_*(a \otimes b)$. Then, for an element $a \in \mathscr{A}(G)^\circ$ group-like or primitive with respect to δ_e, AD_{*a} is a linear map on the Lie superalgebra $\mathfrak{g}$.*

Proof. Consider first the case where a is a group-like element, $a = \delta_g, g \in G$. If $v \in \mathfrak{g}$, we have: $AD_{*a}(v) = AD_*(a \otimes v) = a \odot v \odot a^{-1}$, because for group-like elements the antipode $s^\circ_{\mathscr{A}}$ of $\mathscr{A}(G)^\circ$ is given by $s^\circ_{\mathscr{A}} a = a^{-1} = \delta_{g^{-1}}$. It is then immediate to verify that if $\Delta^\circ_{\mathscr{A}}$ is the coproduct of $\mathscr{A}(G)^\circ$, we have: $\Delta^\circ_{\mathscr{A}}(AD_{*a}(v)) = AD_{*a}(v) \otimes \delta_e + \delta_e \otimes AD_{*a}(v)$ which means that $AD_{*a}(v)$ belongs also to $\mathfrak{g}$. Proceeding in the same way for the case where a is primitive with respect to δ_e, we find $AD_{*a}(v) = a \odot v - (-1)^{|a||v|} v \odot a = [a, v] \in \mathfrak{g}$. Thus, for an element $a \in \mathscr{A}(G)^\circ$ group-like or primitive with respect to δ_e, we take $AD_{*a} \in \operatorname{End} \mathfrak{g}$. ■

The previous proof makes clear that if $a = \delta_g$, $g \in G$, then AD_{*a} is an isomorphism of the Lie superalgebra $\mathfrak{g}$. Indeed, in this case $AD_{*a} = R^{-1}_{g*} \circ L_{g*}$. For the case where a is primitive with respect to δ_e, AD_{*a} coincides with the adjoint representation of $\mathfrak{g}$ on itself, $AD_{*a} = ad(a)$, where $ad(a)(b) = [a, b], \forall b \in \mathfrak{g}$.

3.13 Remark. *One can define a linear map* $\Psi_{*a}\colon \mathscr{B}(Y)^\circ \to \mathscr{B}(Y)^\circ$ *for each left action* $\Psi\colon (G,\mathscr{A}) \times (Y,\mathscr{B}) \to (Y,\mathscr{B})$ *and* $a \in \mathscr{A}(G)^\circ$ *by* $\Psi_{*a}(b) = \Psi_*(a \otimes b)$, $\forall b \in \mathscr{B}(Y)^\circ$. *It is then easily verified that for a group-like,* Ψ_{*a} *is an isomorphism of graded coalgebras; furthermore,* $\Psi_{*a} = \Psi_{g*}$ *if* $a = \delta_g$ *(see Theorem 3.9). We have analogous facts for a right action.*

Graded Lie groups provide an important and wide class of parallelizable graded manifolds as the following theorem asserts.

3.14 Theorem. *Each graded Lie group* $(G,\mathscr{A})$ *is a parallelizable graded manifold.*

Proof. Let $\Phi\colon (G,\mathscr{A}) \times (G,\mathscr{A}) \to (G,\mathscr{A})$ be the right action such that $\Phi^* = \Delta_{\mathscr{A}}$ (see the proof of Corollary 3.10). Then, $(\Phi^*)_a = (id \otimes a) \circ \Phi^* = r_a$. If $a \in \mathfrak{g}$, then we know by Theorem 3.9 that $(\Phi^*)_a$ is a derivation on $\mathscr{A}(G)$. For $g \in G$, the tangent vector $\widetilde{(\Phi^*)}_a(g) \in T_g(G,\mathscr{A})$ is calculated by means of (2.6): $\widetilde{(\Phi^*)}_a(g) = \delta_g \circ (\Phi^*)_a = \delta_g \odot a = L_{g*}(a)$; but L_{g*} is an isomorphism by Proposition 3.5. This means that if $\{a^i, b^j\}$ is a basis of the Lie superalgebra $\mathfrak{g}$, $a^i \in \mathfrak{g}_0, b^j \in \mathfrak{g}_1$, then $\left\{\widetilde{(\Phi^*)}_{a^i}(g), \widetilde{(\Phi^*)}_{b^j}(g)\right\}$ is a basis of $T_g(G,\mathscr{A})$ for each $g \in G$. By Proposition 2.12.1 of [**15**], we conclude that $\{(\Phi^*)_{a^i}, (\Phi^*)_{b^j}\}$ is a global basis of $\mathfrak{Der}\mathscr{A}(G)$ for its left $\mathscr{A}(G)$-module structure. ■

4. Actions, graded distributions and quotient structures

Two important notions in the study of graded Lie group actions on graded manifolds are those of graded distributions and quotient graded manifolds. In this section, we investigate the relation between graded distributions and free actions properly defined in the graded setting. In addition, we find a necessary and sufficient condition in order that the quotient defined by the action of a graded Lie group be a graded manifold.

We first introduce the notion of the graded distribution (see also [**9**]).

4.1 Definition. *Let* $(M,\mathscr{A})$ *be a graded manifold of dimension* (m,n). *We call graded distribution of dimension* (p,q) *on* $(M,\mathscr{A})$ *a sheaf* $U \to \mathscr{E}(U)$ *of free* $\mathscr{A}(U)$*-modules such that each* $\mathscr{E}(U)$ *is a graded submodule of* $\mathfrak{Der}\mathscr{A}(U)$ *of*

dimension (p,q). The distribution will be called involutive if for each $\xi,\eta\in\mathscr{E}(U)$, we have $[\xi,\eta]\in\mathscr{E}(U), \forall U\subset M$ open.

Thus, for each open $U\subset M$, there exist elements $\xi_i\in\big(\mathfrak{Der}\mathscr{A}(U)\big)_0, i=1,\dots,p$, $\eta_j\in\big(\mathfrak{Der}\mathscr{A}(U)\big)_1, j=1,\dots,q$ such that

$$\mathscr{E}(U)=\mathscr{A}(U)\cdot\xi_1\oplus\cdots\oplus\mathscr{A}(U)\cdot\xi_p\oplus\mathscr{A}(U)\cdot\eta_1\oplus\cdots\oplus\mathscr{A}(U)\cdot\eta_q.$$

Given the graded distribution $\mathscr{E}$ on $(M,\mathscr{A})$, one obtains, for each $x\in M$, a graded subspace E_x of $T_x(M,\mathscr{A})$, calculating the tangent vectors $\tilde{\xi}_x\in T_x(M,\mathscr{A})$ via relation (2.6), for each $\xi\in\mathscr{E}(U)$, $x\in U$. Clearly, $E_x=(E_x)_0\oplus(E_x)_1$ with $\dim(E_x)_0=\epsilon_0(x)\le p$ and $\dim(E_x)_1=\epsilon_1(x)\le q$. Therefore, we make the following distinction:

4.2 Definition. *A graded distribution $\mathscr{E}$ of dimension (p,q) on the graded manifold $(M,\mathscr{A})$ is called 0-regular if $\epsilon_0(x)=p$, and 1-regular if $\epsilon_1(x)=q$, for each $x\in M$. We say that $\mathscr{E}$ is regular if $\epsilon_0(x)=p$ and $\epsilon_1(x)=q$, for each $x\in M$.*

For the subsequent analysis, a graded generalization of free actions will be necessary.

4.3 Definition. *We call the right action Φ: $(Y,\mathscr{B})\times(G,\mathscr{A})\to(Y,\mathscr{B})$ free, if for each $y\in Y$ the morphism Φ_y: $(G,\mathscr{A})\to(Y,\mathscr{B})$ is such that Φ_{y*}: $\mathscr{A}(G)^\circ\to\mathscr{B}(Y)^\circ$ is injective.*

Similarly, one defines the left free action. It is clear that if the graded Lie group $(G,\mathscr{A})$ acts freely on $(Y,\mathscr{B})$, then we obtain a free action of G on Y, but if only the restriction $\Phi_*|_{Y\times G}$ is a free action then, in general, the action Φ is not free.

An equivalent characterization of the free action in graded Lie theory is provided by the following proposition for the case of a right action.

4.4 Proposition. *The action Φ: $(Y,\mathscr{B})\times(G,\mathscr{A})\to(Y,\mathscr{B})$ is free if and only if the morphism of graded manifolds $\tilde{\Phi}=(\Phi\times\pi_1)\circ\Delta$: $(Y,\mathscr{B})\times(G,\mathscr{A})\to(Y,\mathscr{B})\times(Y,\mathscr{B})$ is such that $\tilde{\Phi}_*$ is injective. Here, Δ denotes the diagonal morphism on $(Y,\mathscr{B})\times(G,\mathscr{A})$ and π_1 is the projection on the first factor.*

Proof. Consider elements $a = \delta_g \in \mathscr{A}(G)^\circ, b = \delta_y \in \mathscr{B}(Y)^\circ$ group-like and $u \in T_y(Y, \mathscr{B}), w \in T_g(G, \mathscr{A})$ primitive. Then, a simple calculation gives

$$\tilde{\Phi}_*(b \otimes a) = \Phi_{y*}(a) \otimes b \tag{4.1}$$

$$\tilde{\Phi}_*(u \otimes a + b \otimes w) = [\Phi_{g*}(u) + \Phi_{y*}(w)] \otimes b + \Phi_{g*}(b) \otimes u. \tag{4.2}$$

Suppose now that Φ is a free action; then the morphism $\Phi_{y*}\colon \mathscr{A}(G)^\circ \to \mathscr{B}(Y)^\circ$ is injective which implies immediately, thanks to (4.1) and (4.2), that $\tilde{\Phi}_*$ is injective on all group-like and primitive elements. By Proposition 2.17.1 of [**15**], this is a necessary and sufficient condition for the morphism $\tilde{\Phi}_*$ to be injective on the whole graded coalgebra $\mathscr{A}(G)^\circ$. The converse is immediate again by (4.1) and (4.2). ■

Consider now a right action $\Phi\colon (Y, \mathscr{B}) \times (G, \mathscr{A}) \to (Y, \mathscr{B})$; by Theorem 3.9, we have a linear map $\boldsymbol{I}_\Phi\colon \mathfrak{g} \to \mathfrak{Der}\mathscr{B}(Y)$ defined as $\boldsymbol{I}_\Phi(a) = (\Phi^*)_a$. We thus obtain a subspace $\mathfrak{Der}_\Phi\mathscr{B}(Y) = \operatorname{im}\boldsymbol{I}_\Phi$ of the Lie superalgebra of derivations on $\mathscr{B}(Y)$. As a matter of fact, $\mathfrak{Der}_\Phi\mathscr{B}(Y)$ is a graded Lie subalgebra of $\mathfrak{Der}\mathscr{B}(Y)$. Indeed, one readily verifies that for all $a, b \in \mathfrak{g}$ we have $[(\Phi^*)_a, (\Phi_*)_b] = (\Phi^*)_{[a,b]}$, which means that $\boldsymbol{I}_\Phi([a, b]) = [\boldsymbol{I}_\Phi(a), \boldsymbol{I}_\Phi(b)]$. The following theorem provides an important property of free actions on graded manifolds.

4.5 Theorem. *Let Φ be a free right action of the graded Lie group $(G, \mathscr{A})$ on the graded manifold $(Y, \mathscr{B})$, $\dim(G, \mathscr{A}) = (m, n)$. Then Φ induces a regular and involutive graded distribution $\mathscr{E}$ on $(Y, \mathscr{B})$ of dimension (m, n).*

Proof.

• *Step 1.* Let us first calculate the kernel of the Lie superalgebra morphism $\boldsymbol{I}_\Phi\colon \mathfrak{g} \to \mathfrak{Der}\mathscr{B}(Y)$ when Φ is a free action. To this end, the following general property of actions is useful:

$$\widetilde{(\Phi^*)}_a(y) = \Phi_{y*}(a),\ \forall y \in Y,\ \forall a \in \mathfrak{g}. \tag{4.3}$$

For the proof of (4.3), we note only that, by relation (2.6), $\widetilde{(\Phi^*)}_a(y) = \delta_y \circ (\Phi^*)_a = \Phi_*(\delta_y \otimes a)$. Suppose now that $\boldsymbol{I}_\Phi(a) = 0 \Leftrightarrow (\Phi^*)_a = 0$; by (4.3), this implies that $\Phi_{y*}(a) = 0, \forall y \in Y$. Since Φ is a free action, we know by Definition 4.3, that Φ_{y*} is injective for all $y \in Y$, which implies that $a = 0$. As a result, $\ker\boldsymbol{I}_\Phi = 0$, or $\boldsymbol{I}_\Phi$ is

injective; hence, $\mathfrak{Der}_\Phi\mathscr{B}(Y)$ is a graded Lie subalgebra of $\mathfrak{Der}\mathscr{B}(Y)$ whose even and odd dimensions are m and n respectively: $(\mathfrak{Der}_\Phi\mathscr{B}(Y))_0 \cong \mathfrak{g}_0$, $(\mathfrak{Der}_\Phi\mathscr{B}(Y))_1 \cong \mathfrak{g}_1$.

• *Step 2.* Let now $\mathfrak{Der}_\Phi\mathscr{B}$ be the correspondence $U \to \mathfrak{Der}_\Phi\mathscr{B}(U)$, where $\mathfrak{Der}_\Phi\mathscr{B}(U) = P_{YU}\big(\mathfrak{Der}_\Phi\mathscr{B}(Y)\big)$, $U \subset Y$ and $P_{UV}\colon \mathfrak{Der}\mathscr{B}(U) \to \mathfrak{Der}\mathscr{B}(V)$ are the restriction maps for the sheaf $\mathfrak{Der}\mathscr{B}$. Clearly, $\mathfrak{Der}_\Phi\mathscr{B}$ is a subpresheaf of $\mathfrak{Der}\mathscr{B}$. Consider now the subpresheaf $\mathscr{E} = \mathscr{B} \cdot \mathfrak{Der}_\Phi\mathscr{B}$ of $\mathfrak{Der}\mathscr{B}$, $\mathscr{E}(U) = \mathscr{B}(U) \cdot \mathfrak{Der}_\Phi\mathscr{B}(U) = P_{YU}\big(\mathscr{B}(Y) \cdot \mathfrak{Der}_\Phi\mathscr{B}(Y)\big)$. $\mathscr{E}(U)$ is the set of finite linear combinations of elements of $\mathfrak{Der}_\Phi\mathscr{B}(U)$ with coefficients in $\mathscr{B}(U)$. In order to prove that $\mathscr{E}$ is a sheaf, let us consider an open $U \subset Y$, an open covering $\{U_\alpha\}_{\alpha\in\Lambda}$ of U and elements $D_\alpha \in \mathscr{E}(U_\alpha)$ such that $P_{U_\alpha U_{\alpha\beta}}(D_\alpha) = P_{U_\beta U_{\alpha\beta}}(D_\beta)$, $\forall \alpha,\beta \in \Lambda$ when $U_{\alpha\beta} = U_\alpha \cap U_\beta \neq \varnothing$. Then by the sheaf properties of $\mathfrak{Der}\mathscr{B}$, there exists an element $D \in \mathfrak{Der}\mathscr{B}(U)$ such that $P_{UU_\alpha}(D) = D_\alpha$. But if we write $D_\alpha = \sum_i f^i_\alpha P_{YU_\alpha}(\Phi^*)_{e_i}$, $f^i_\alpha \in \mathscr{B}(U_\alpha)$ and $\{e_i\}$ is a basis of $\mathfrak{g}$, then by step 1, we find easily that $f^i_\alpha = f^i|_{U_\alpha}$, $f^i \in \mathscr{B}(U)$, because D_α's coincide on the intersections $U_{\alpha\beta}$. This means that $D_\alpha = P_{YU_\alpha}(E)$, where $E = \sum_i F^i(\Phi^*)_{e_i}$ with $F^i \in \mathscr{B}(Y)$ such that $F^i|_U = f^i$, $\forall i$. It is then immediate that $D = P_{YU}(E) \in \mathscr{E}(U)$.

• *Step 3.* It is evident that the sheaf $\mathscr{E}$ previously constructed, has the properties of a graded distribution. In fact, $\mathscr{E}(U)$ is a graded submodule of $\mathfrak{Der}\mathscr{B}(U)$ of dimension (m,n) for each open $U \subset Y$. This distribution is clearly regular thanks to relation (4.3) and to the fact that the action is free. It remains to show that it is involutive. To this end, consider two elements $\xi = \sum_i f^i P_{YU}(\Phi^*)_{a^i}$ and $\eta = \sum_j g^j P_{YU}(\Phi^*)_{b^j}$ of $\mathscr{E}(U)$, with $f^i, g^j \in \mathscr{B}(U), a^i, b^j \in \mathfrak{g}$. Then, direct calculation shows that

$$\begin{aligned}
[\xi,\eta] = {} & \sum_{i,j} f^i \big(P_{YU}(\Phi^*)_{a^i} g^j\big) P_{YU}(\Phi^*)_{b^j} \\
& - (-1)^{|\xi||\eta|} \sum_{i,j} g^j \big(P_{YU}(\Phi^*)_{b^j} f^i\big) P_{YU}(\Phi^*)_{a^i} \\
& + \sum_{i,j} (-1)^{|a^i||g^j|} f^i g^j P_{YU}(\Phi^*)_{[a^i,b^j]},
\end{aligned}$$

from which the involutivity is evident. ■

We focus now our attention on graded quotient structures defined by equivalence relations on graded manifolds (see [**1**] for a general treatment on this subject).

A special case of equivalence relation is provided by the action of a graded Lie group on a graded manifold and this will be the interesting one for us.

4.6 Definition. *We call a right action* $\Phi\colon (Y,\mathscr{B}) \times (G,\mathscr{A}) \to (Y,\mathscr{B})$ *regular if the morphism* $\tilde{\Phi} = (\Phi \times \pi_1) \circ \Delta\colon (Y,\mathscr{B}) \times (G,\mathscr{A}) \to (Y,\mathscr{B}) \times (Y,\mathscr{B})$ *defines* $(Y,\mathscr{B}) \times (G,\mathscr{A})$ *as a closed graded submanifold of* $(Y,\mathscr{B}) \times (Y,\mathscr{B})$.

Recall here, [**15**], that $(R,\mathscr{D})$ is a graded submanifold of $(Y,\mathscr{B})$ if $\mathscr{D}(R)^\circ \subset \mathscr{B}(Y)^\circ$ and there exists a morphism of graded manifolds $i\colon (R,\mathscr{D}) \to (Y,\mathscr{B})$ such that $i_*\colon \mathscr{D}(R)^\circ \to \mathscr{B}(Y)^\circ$ is simply the inclusion; $(R,\mathscr{D})$ will be called closed if, furthermore, $\dim(R,\mathscr{D}) < \dim(Y,\mathscr{B})$. Then, the action is regular if the subset $\tilde{\Phi}_*\big(\mathscr{B}(Y)^\circ \otimes \mathscr{A}(G)^\circ\big) \subset \mathscr{B}(Y)^\circ \otimes \mathscr{B}(Y)^\circ$ defines a graded submanifold of $(Y,\mathscr{B}) \times (Y,\mathscr{B})$ in the sense of Kostant, [**15**]. The following theorem generalizes in a natural way to the graded case, a fundamental result about quotients defined by actions in ordinary manifold theory.

4.7 Theorem. *The action* $\Phi\colon (G,\mathscr{A}) \times (Y,\mathscr{B}) \to (Y,\mathscr{B})$ *is regular if and only if the quotient* $(Y/G,\mathscr{B}/\mathscr{A})$ *is a graded manifold.*

Proof. Thanks to Theorem 2.6 of [**1**], it suffices to prove that the projections $p_i\colon (Y,\mathscr{B}) \times (Y,\mathscr{B}) \to (Y,\mathscr{B}), i = 1,2$ on the first and second factors restricted to the image of $(Y,\mathscr{B}) \times (G,\mathscr{A})$ under $\tilde{\Phi}$ are submersions. In other words, we must show that the morphisms of graded coalgebras $p_{i*} \circ \tilde{\Phi}_*\colon \mathscr{B}(Y)^\circ \otimes \mathscr{A}(G)^\circ \to \mathscr{B}(Y)^\circ, i = 1,2$ restricted to primitive elements are surjective.

Consider an arbitrary primitive element $V = u \otimes a + b \otimes w$, for $a = \delta_g \in \mathscr{A}(G)^\circ, b = \delta_y \in \mathscr{B}(Y)^\circ$ group-like and $u \in T_y(Y,\mathscr{B})$, $w \in T_g(G,\mathscr{A})$ primitive. Using relation (4.2), we find easily: $p_{1*}\tilde{\Phi}_*(V) = \Phi_{g*}(u) + \Phi_{y*}(w)$ and $p_{2*}\tilde{\Phi}_*(V) = u$, which proves that $p_i \circ \tilde{\Phi}$ are submersions, $i = 1,2$. ■

Note here that if $U \subset Y/G$ is an open subset, then the sheaf $\mathscr{B}/\mathscr{A}$ is given by

$$(\mathscr{B}/\mathscr{A})(U) = \{f \in \mathscr{B}(\breve{\pi}^{-1}(U)) \mid \Phi^* f = f \otimes 1\!\!1_{\mathscr{A}}\}, \tag{4.4}$$

where $\breve{\pi}\colon Y \to Y/G$ is the projection, [**1**]. Furthermore, the dimension of the quotient graded manifold $(Y/G,\mathscr{B}/\mathscr{A})$ is equal to $\dim(Y/G,\mathscr{B}/\mathscr{A}) = 2\dim(Y,\mathscr{B}) - \dim(\operatorname{im}\tilde{\Phi})$, where $\operatorname{im}\tilde{\Phi}$ denotes the closed graded submanifold defined by $\tilde{\Phi}$. When

Φ is a free action, then by Proposition 4.4, we take that $\dim(Y/G, \mathscr{B}/\mathscr{A}) = \dim(Y, \mathscr{B}) - \dim(G, \mathscr{A})$.

We make finally some comments about graded isotropy subgroups recalling their construction from [**15**], but in a more concise way. Consider a right action $\Phi\colon (Y, \mathscr{B}) \times (G, \mathscr{A}) \to (Y, \mathscr{B})$ and $b \in \mathscr{B}(Y)^\circ$ a group-like element, $b = \delta_y$, $y \in Y$. Let $H_y(G, \mathfrak{g})$ be the set of elements $a \in \mathscr{A}(G)^\circ$ with the property

$$\Phi_*(\delta_y \otimes a) = \epsilon^\circ_{\mathscr{A}}(a)\delta_y. \tag{4.5}$$

Let $H_y(G, \mathfrak{g}) \cap G = G_y$ and $H_y(G, \mathfrak{g}) \cap \mathfrak{g} = \mathfrak{g}_y$, then $(\mathfrak{g}_y)_0$ is the Lie algebra of G_y. It is then clear that we can form the Lie-Hopf algebra $\mathbf{R}(G_y) \circledast E(\mathfrak{g}_y)$ because $\mathfrak{g}_y$ is stable under the adjoint action of G_y, see Proposition 3.12 and relation (4.5). By Proposition 3.8.3 of [**15**], $\mathbf{R}(G_y) \circledast E(\mathfrak{g}_y)$ corresponds to a graded Lie subgroup of $(G, \mathscr{A})$. We denote this subgroup by $(G_y, \mathscr{A}_y)$ and call it graded isotropy subgroup of $(G, \mathscr{A})$ at the point y.

5. Graded principal bundles

Graded principal bundles were first introduced in [**1**], [**2**]. Here, we discuss this notion with slight modifications suggested by the requirement that the definition of graded principal bundles reproduces well the ordinary principal bundles.

5.1 Definition. *A graded principal bundle over a graded manifold $(X, \mathscr{C})$ consists of a graded manifold $(Y, \mathscr{B})$ and an action Φ of a graded Lie group $(G, \mathscr{A})$ on $(Y, \mathscr{B})$ with the following properties:*

1. *Φ is a free right action*
2. *the quotient $(Y/G, \mathscr{B}/\mathscr{A})$ is a graded manifold, isomorphic to $(X, \mathscr{C})$, such that the natural projection $\pi\colon (Y, \mathscr{B}) \to (X, \mathscr{C})$ is a submersion*
3. *$(Y, \mathscr{B})$ is locally trivial that is, for each open $U \subset X$, there exists an isomorphism of graded manifolds $\phi\colon (V, \mathscr{B}|_V) \to (U \times G, \mathscr{C}|_U \hat{\otimes}_\pi \mathscr{A})$, $V = \pi_*^{-1}(U) \subset Y$, such that the isomorphism ϕ^* of graded algebras is a morphism of $\mathscr{A}(G)$-comodules, where the $\mathscr{A}(G)$-comodule structures on $\mathscr{C}(U) \hat{\otimes}_\pi \mathscr{A}(G)$*

and $\mathscr{B}(V)$ are given by $id \otimes \Delta_{\mathscr{A}}$ and Φ^* respectively. Furthermore, we require that $\phi^* = m_{\mathscr{B}} \circ (\pi^* \otimes \psi^*)$, where ψ: $(V, \mathscr{B}|_V) \to (G, \mathscr{A})$ is a morphism of graded manifolds.

The fact that ϕ^* is a morphism of $\mathscr{A}(G)$-comodules, that is,

$$(\phi^* \otimes id) \circ (id \otimes \Delta_{\mathscr{A}}) = \Phi^* \circ \phi^* \tag{5.1}$$

implies that ψ^* is also a morphism of $\mathscr{A}(G)$-comodules:

$$(\psi^* \otimes id) \circ \Delta_{\mathscr{A}} = \Phi^* \circ \psi^*. \tag{5.2}$$

One easily verifies that the underlying differentiable manifolds of Definition 5.1 form an ordinary principal bundle, and further, if the graded manifolds become trivial, in the sense that $\mathscr{A} = C^\infty_G, \mathscr{B} = C^\infty_Y, \mathscr{C} = C^\infty_X$, then we obtain the definition of an ordinary principal bundle. We refer the reader to [**14**] for a general and systematic treatment on the subject of principal bundles in differential geometry.

Let us now compute the graded isotropy subgroups in the case where the action Φ is free (for example, this is the case of the graded principal bundle). The Lie group G_y which is defined as $G_y = \{g \in G \mid \Phi_*(b \otimes \delta_g) = b\}$ is equal to e because the action of G on Y is free. On the other hand, $\mathfrak{g}_y = \{a \in \mathfrak{g} \mid \Phi_*(b \otimes a) = 0\} = 0$, again because the action is free (the morphism Φ_{y*} is injective). Hence, in this case the graded isotropy subgroup $(G_y, \mathscr{A}_y)$ is simply $(e, \mathscr{R})$, where $\mathscr{R}$ is the trivial sheaf over the identity $e \in G$, $\mathscr{R}(e) = \mathbf{R}$.

In order to calculate the quotient graded manifold $(G/G_y, \mathscr{A}/\mathscr{A}_y)$ which represents the orbit of the point y under the action of $(G, \mathscr{A})$, we need the expression of the canonical right action Φ of the subgroup $(G_y, \mathscr{A}_y)$ on $(G, \mathscr{A})$. If i: $(G_y, \mathscr{A}_y) \to (G, \mathscr{A})$ is the inclusion, we have: $\Phi^* = (id \otimes i^*) \circ \Delta_{\mathscr{A}}$ and for the case of the free action, where $(G_y, \mathscr{A}_y) = (e, \mathscr{R})$, one finds that $i^* = \epsilon_{\mathscr{A}}$ and finally $\Phi^* = id$. In view of relation (4.4), it is straightforward that $\mathscr{A}/\mathscr{A}_y = \mathscr{A}$. Therefore,

5.2 Property. *The orbits of a free action of $(G, \mathscr{A})$ are always isomorphic as graded manifolds to $(G, \mathscr{A})$.*

For the case now of the graded principal bundle, the orbit $(\mathscr{O}_y, \mathscr{B}_y)$ of $(G, \mathscr{A})$ through $y \in Y$ will be called fibre of $(Y, \mathscr{B})$ over $x = \pi_*|_Y(y) \in X$. Using the graded version of the submersion theorem, [**23**], one can justify this terminology as follows. The pre-image $\pi^{-1}(x, \mathscr{R})$ of the closed graded submanifold $(x, \mathscr{R}) \hookrightarrow (X, \mathscr{C})$ is a closed graded submanifold of $(Y, \mathscr{B})$ whose underlying differentiable manifold is $\pi_*^{-1}(x)$. So, if we write $\pi^{-1}(x, \mathscr{R}) = (\pi_*^{-1}(x), \mathscr{D})$, then for each $z \in \pi_*^{-1}(x)$ we have: $T_z(\pi_*^{-1}(x), \mathscr{D}) = \ker T_z\pi$. We know already that $\pi_*^{-1}(x) = \mathscr{O}_y$, the orbit under G of a point $y \in Y$ such that $\pi_*|_Y(y) = x$. Furthermore, if $\delta_z = \Phi_*(\delta_y \otimes \delta_g)$, $g \in G$ and $v \in T_g(G, \mathscr{A})$, then $V = \Phi_*(\delta_y \otimes v) \in T_z(\mathscr{O}_y, \mathscr{B}_y)$ and $\pi_*(V) = 0$. Consequently, $T_z(\mathscr{O}_y, \mathscr{B}_y) \subset \ker T_z\pi$. By a simple argument on dimensions, we obtain that $T_z(\mathscr{O}_y, \mathscr{B}_y) = T_z(\pi_*^{-1}(x), \mathscr{D})$. We conclude that the tangent bundles of $\pi^{-1}(x, \mathscr{R})$ and $(\mathscr{O}_y, \mathscr{B}_y)$ are identical; but then, Theorem 2.16 of [**15**] tells us that these graded manifolds coincide.

Next, we discuss an elementary example of graded principal bundle, the product bundle. In this case, one can directly verify the axioms of Definition 5.1. Nevertheless, there exist also graded principal bundles for which Definition 5.1 cannot be directly applied, even though this is possible for the corresponding ordinary principal bundles. For such cases, one may use an equivalent definition of the graded principal bundle, see next section.

5.3 Example. Consider a graded manifold $(X, \mathscr{C})$, a graded Lie group $(G, \mathscr{A})$ and their product $(Y, \mathscr{B}) = (X, \mathscr{C}) \times (G, \mathscr{A})$. One has a canonical right action Φ: $(Y, \mathscr{B}) \times (G, \mathscr{A}) \to (Y, \mathscr{B})$ defined as $\Phi^* = id \otimes \Delta_{\mathscr{A}}$. This action is free: if $\delta_y = \delta_x \otimes \delta_g \in \mathscr{C}(X)^\circ \otimes \mathscr{A}(G)^\circ$ is group-like and $a \in \mathscr{A}(G)^\circ$, then $\Phi_{y*}(a) = \Phi_*(\delta_x \otimes \delta_g \otimes a) = \delta_x \otimes (\delta_g \odot a) = \delta_x \otimes L_{g*}(a)$, which implies that Φ_{y*} is an injective morphism of graded coalgebras, because L_{g*} is an isomorphism. Evidently, the quotient Y/G is equal to X and the sheaf $\mathscr{B}/\mathscr{A}$ over X is given by the elements $f \in \mathscr{C}(U) \hat{\otimes}_\pi \mathscr{A}(G)$ for which $\Phi^* f = f \otimes \mathbb{1}_{\mathscr{A}}$, $U \subset X$ open. If we decompose f as $f = \sum_i f_i \otimes h_i$, $f_i \in \mathscr{C}(U), h_i \in \mathscr{A}(G)$, we take easily $\Delta_{\mathscr{A}} h_i = h_i \otimes \mathbb{1}_{\mathscr{A}}$, hence h_i's are such that $\epsilon_{\mathscr{A}}(h_i)\mathbb{1}_{\mathscr{A}} = h_i$. We conclude that f is of the form $f = \sum_i \epsilon_{\mathscr{A}}(h_i) f_i \otimes \mathbb{1}_{\mathscr{A}} = f_{\mathscr{C}} \otimes \mathbb{1}_{\mathscr{A}}$ and finally $(\mathscr{B}/\mathscr{A})(U) \cong \mathscr{C}(U)$, which proves that the quotient $(Y/G, \mathscr{B}/\mathscr{A})$ is isomorphic to $(X, \mathscr{C})$. Further, the identity map $\phi^* = id$: $\mathscr{C}(U) \hat{\otimes}_\pi \mathscr{A}(G) \to \mathscr{C}(U) \hat{\otimes}_\pi \mathscr{A}(G)$ admits the decomposi-

tion of Definition 5.1 (it suffices to choose $\pi = \pi_1, \psi = \pi_2$, the projections on the first and second factors respectively) and satisfies trivially the relation (5.1).

6. The geometry of graded principal bundles

In this section we analyze three aspects of the geometry of graded principal bundles: the relation between the sheaf of vertical derivations and the graded distribution induced by the action of the structure group, a criterion of global triviality of the graded principal bundle, and, finally, a way to reformulate Definition 5.1 avoiding the use of local trivializations.

For this and the subsequent sections, we will adopt the following notation in order to simplify the discussion: if $a \in \mathfrak{g}$ and $V \subset Y$ is an open, then the restriction $P_{YV}(\Phi^*)_a$ (Theorem 4.5) will be simply denoted by $(\Phi^*)_a$.

It is well-known that if $Y(X, G)$ is an ordinary principal bundle, then the set of vertical vectors at $y \in Y$ is equal to the set of induced vectors at the same point. In the previous section, we saw that the same is true for graded principal bundles; however, it is not evident that this property remains valid for the sheaves of vertical and induced derivations. Nonetheless, as the following theorem confirms, this is indeed the case.

6.1 Theorem. *Let $(Y, \mathscr{B})$ be a graded principal bundle over $(X, \mathscr{C})$ with structure group $(G, \mathscr{A})$. If $\mathscr{E}$ is the natural graded distribution induced by the free action of $(G, \mathscr{A})$ on $(Y, \mathscr{B})$, then $\mathscr{E}$ is equal to the sheaf of vertical derivations, $\mathscr{E} = \mathscr{V}\!er(\pi_*, \mathscr{B})$.*

Proof. We show first that $\mathscr{E} \subset \mathscr{V}\!er(\pi_*, \mathscr{B})$; to this end, it is sufficient to prove that for each $a \in \mathfrak{g}$, we have $\pi_*(\Phi^*)_a = 0$. Indeed, if f is a homogeneous element of $\mathscr{C}(U)$, $U \subset X$ open, we take:

$$\pi^*\left[\pi_*(\Phi^*)_a(f)\right] = (\Phi^*)_a(\pi^* f) = (id \otimes a)(\pi^* f \otimes \mathbb{1}_{\mathscr{A}}) = 0,$$

since a is primitive with respect to δ_e. Now the following argument on dimensions completes the proof. A derivation $D \in \mathscr{V}\!er(\pi_*, \mathscr{B})(V)$, $V = \pi_*^{-1}(U)$, is characterized by the property: $\pi^*[(\pi_* D)f] = D(\pi^* f) = 0$, $\forall f \in \mathscr{C}(U)$. This means

in terms of coordinates that D does not depend on the graded coordinates on V obtained by pulling-back the graded coordinates of U via π^*. Using the fact that π^* is an injection, we find that the dimension of $\mathscr{V}\!er(\pi_*, \mathscr{B})(V)$ equals to $\dim(Y, \mathscr{B}) - \dim(X, \mathscr{C}) = \dim(G, \mathscr{A}) = \dim \mathscr{E}(V)$. ∎

The fact that a local trivialization ϕ is an isomorphism of $\mathscr{A}(G)$-comodules is expressed by relation (5.1) but it is also reflected in the induced derivations. The following lemma makes this precise, providing a relation between them.

6.2 Lemma. *If ϕ: $(V, \mathscr{B}|_V) \to (U, \mathscr{C}|_U) \times (G, \mathscr{A})$, $V = \pi_*^{-1}(U)$, is a local trivialization of $(Y, \mathscr{B})$, then the following relation is true for each $a \in \mathfrak{g}$:*

$$\phi_*(\Phi^*)_a = id \otimes (R^*)_a.$$

Proof. We show first that $\phi_*(\Phi^*)_a \in \mathfrak{Der}\mathscr{A}(G)$. Indeed, if $f_{\mathscr{C}} \in \mathscr{C}(U)$, we take:

$$\phi_*(\Phi^*)_a(f_{\mathscr{C}} \otimes \mathbb{1}_{\mathscr{A}}) = \big((\phi^{-1})^* \circ (\Phi^*)_a \circ \phi^*\big)(f_{\mathscr{C}} \otimes \mathbb{1}_{\mathscr{A}}) = (\phi^*)^{-1}(\Phi^*)_a(\pi^* f_{\mathscr{C}}) = 0.$$

It is then sufficient to calculate $\phi_*(\Phi^*)_a$ on elements of $\mathscr{C}(U) \hat{\otimes}_\pi \mathscr{A}(G)$ of the form $\mathbb{1}_{\mathscr{C}} \otimes f_{\mathscr{A}}$ for $f_{\mathscr{A}} \in \mathscr{A}(G)$. Taking into account relation (5.2) and if $\Delta_{\mathscr{A}} f_{\mathscr{A}} = \sum_i I^i f_{\mathscr{A}} \otimes J^i f_{\mathscr{A}}$, one finds:

$$\begin{aligned}
\phi_*(\Phi^*)_a(\mathbb{1}_{\mathscr{C}} \otimes f_{\mathscr{A}}) &= (\phi^{-1})^*(id \otimes a)(\psi^* \otimes id)\Delta_{\mathscr{A}} f_{\mathscr{A}} \\
&= \sum_i (-1)^{|a||I^i f_{\mathscr{A}}|}(\phi^*)^{-1}\psi^*(I^i f_{\mathscr{A}})a(J^i f_{\mathscr{A}}) \\
&= \sum_i (-1)^{|a||I^i f_{\mathscr{A}}|}(\mathbb{1}_{\mathscr{C}} \otimes I^i f_{\mathscr{A}})a(J^i f_{\mathscr{A}}) \\
&= \mathbb{1}_{\mathscr{C}} \otimes (id \otimes a)\Delta_{\mathscr{A}} f_{\mathscr{A}} = (id \otimes (R^*)_a)(\mathbb{1}_{\mathscr{C}} \otimes f_{\mathscr{A}}),
\end{aligned}$$

where we have used that $\phi^*(\mathbb{1}_{\mathscr{C}} \otimes I^i f_{\mathscr{A}}) = \psi^*(I^i f_{\mathscr{A}})$ implies $(\phi^{-1})^*\psi^*(I^i f_{\mathscr{A}}) = \mathbb{1}_{\mathscr{C}} \otimes I^i f_{\mathscr{A}}$. ∎

Next we discuss the notion of section of a graded principal bundle and we show that graded and ordinary sections exhibit several analogous properties.

6.3 Definition. *Let $U \subset X$ be an open on the base manifold $(X, \mathscr{C})$ of a graded principal bundle $(Y, \mathscr{B})$. We call graded section of $(Y, \mathscr{B})$ on U a morphism*

of graded manifolds $s\colon (U, \mathscr{C}|_U) \to (Y, \mathscr{B})$ *having the property* $s^* \circ \pi^* = id$. *We write also* $\pi \circ s = id\colon (U, \mathscr{C}|_U) \to (U, \mathscr{C}|_U)$.

A first property of graded sections is that to each local trivialization, one can associate in a canonical way a graded section. More precisely:

6.4 Lemma. *Let* $\phi\colon (V, \mathscr{B}|_V) \to (U, \mathscr{C}|_U) \times (G, \mathscr{A})$, $V = \pi_*^{-1}(U)$, *be a local trivialization. Then, if* $E\colon \mathscr{C}(U) \mathbin{\hat{\otimes}_\pi} \mathscr{A}(G) \to \mathscr{C}(U)$ *is defined as* $E(f_{\mathscr{C}} \otimes f_{\mathscr{A}}) = \delta_e(f_{\mathscr{A}}) f_{\mathscr{C}}$, *the map* $E \circ (\phi^{-1})^*\colon \mathscr{B}(V) \to \mathscr{C}(U)$ *defines a morphism of graded manifolds with the properties of a graded section.*

Proof. The fact that E is a morphism of graded manifolds is evident because we may write $E = id \otimes \delta_e$. Therefore, there exists a morphism of graded manifolds $s\colon (U, \mathscr{C}|_U) \to (Y, \mathscr{B})$ such that $s^* = E \circ (\phi^{-1})^*$. Now if $f_{\mathscr{C}} \in \mathscr{C}(U)$, we have: $(s^* \circ \pi^*)(f_{\mathscr{C}}) = E\big((\phi^{-1})^* \pi^* f_{\mathscr{C}}\big)$ and since $\phi^*(f_{\mathscr{C}} \otimes \mathbb{1}_{\mathscr{A}}) = \pi^* f_{\mathscr{C}}$, we finally obtain $s^* \circ \pi^* = id$. ■

Conversely now, consider a graded section $s\colon (U, \mathscr{C}|_U) \to (Y, \mathscr{B})$. We wish to show that there exists a local trivialization $\phi\colon (V, \mathscr{B}|_V) \to (U, \mathscr{C}|_U) \times (G, \mathscr{A})$ canonically associated to s, $V = \pi_*^{-1}(U) \subset Y$. To this end, we first define a morphism of graded algebras $\tilde{\phi}^*\colon \mathscr{B}(V) \to \mathscr{C}(U) \mathbin{\hat{\otimes}_\pi} \mathscr{A}(G)$ by $\tilde{\phi}^* = (s^* \otimes id) \circ \Phi^*$. Let $\tilde{\phi}_* = \Phi_* \circ (s_* \otimes id)\colon \mathscr{C}(U)^\circ \otimes \mathscr{A}(G)^\circ \to \mathscr{B}(V)^\circ$ be the corresponding morphism of graded coalgebras. Clearly, the differentiable mapping $\tilde{\phi}_*|_{U \times G}\colon U \times G \to V$ is bijective. Consider now the tangent of ϕ at the arbitrary point $(x, g) \in U \times G$. If $z = u \otimes \delta_g + \delta_x \otimes w \in T_{(x,g)}(U \times G, \mathscr{C}|_U \mathbin{\hat{\otimes}_\pi} \mathscr{A})$ is a tangent vector at (x, g), $u \in T_x(U, \mathscr{C}|_U)$, $w = T_g(G, \mathscr{A})$, then:

$$\tilde{\phi}_*(z) = \Phi_{g*}(s_* u) + (\Phi_{s_* x})_*(w).$$

This implies that $T_{(x,g)}\tilde{\phi}$ is injective, because Φ_{g*} is an isomorphism and Φ_{y*} is injective for each $y \in Y$ (the action is free). Thus, $T_{(x,g)}\tilde{\phi}$ is an injection between two vector spaces of the same dimension and hence an isomorphism. Using now Theorem 2.16 of [**15**], we conclude that $\tilde{\phi}$ is an isomorphism of graded manifolds.

Let now $\pi_1\colon (U, \mathscr{C}|_U) \times (G, \mathscr{A}) \to (U, \mathscr{C}|_U)$ and $\pi_2\colon (U, \mathscr{C}|_U) \times (G, \mathscr{A}) \to (G, \mathscr{A})$ be the projections. Then:

6.5 Proposition. *Let* $\phi\colon (V, \mathscr{B}|_V) \to (U, \mathscr{C}|_U) \times (G, \mathscr{A})$, $V = \pi_*^{-1}(U)$, *be the morphism of graded manifolds defined as* $\phi^* = m_{\mathscr{B}} \circ (\pi^* \otimes \psi^*)$, *where* $\psi^* = (\tilde{\phi}^*)^{-1} \circ \pi_2^*$. *Then,* ϕ *is a local trivialization of the graded principal bundle* $(Y, \mathscr{B})$.

Proof. We show first that ϕ^* is an isomorphism. To this end, consider the composition $\tilde{\phi}^* \circ \phi^*$:

$$\begin{aligned}
\tilde{\phi}^* \circ \phi^* &= (s^* \otimes id) \circ \Phi^* \circ m_{\mathscr{B}} \circ (\pi^* \otimes \psi^*) \\
&= m_{\mathscr{C}\mathscr{A}} \circ [(s^* \otimes id) \otimes (s^* \otimes id)] \circ (\Phi^* \otimes \Phi^*) \circ (\pi^* \otimes \psi^*) \\
&= m_{\mathscr{C}\mathscr{A}} \circ [(s^* \otimes id) \circ \Phi^* \circ \pi^* \otimes (s^* \otimes id) \circ \Phi^* \circ \psi^*] \\
&= m_{\mathscr{C}\mathscr{A}} \circ [(s^* \otimes id) \circ \Phi^* \circ \pi^* \otimes \pi_2^*].
\end{aligned}$$

Using now the fact that $(s^* \otimes id) \circ \Phi^* \circ \pi^* = \pi_1^*$, we find that $\tilde{\phi}^* \circ \phi^* = id$, which proves that ϕ^* is also an isomorphism. It remains to show relation (5.2), or equivalently, $\psi_* \circ \Phi_* = m_{\mathscr{A}}^\circ \circ (\psi_* \otimes id)$. Since $\tilde{\phi}_*$ is an isomorphism between $\mathscr{B}(V)^\circ$ and $\mathscr{C}(U)^\circ \otimes \mathscr{A}(G)^\circ$, we may write each element $b \in \mathscr{B}(V)^\circ$ as $b = \Phi_*(\sum_i s_* c^i \otimes a_0^i)$, for $c^i \in \mathscr{C}(U)^\circ$ and $a_0^i \in \mathscr{A}(G)^\circ$. If $a \in \mathscr{A}(G)^\circ$, we take:

$$\begin{aligned}
\psi_*\Phi_*(b \otimes a) &= \psi_*\Phi_*\big(\Phi_*(\sum_i s_* c^i \otimes a_0^i) \otimes a\big) = \psi_*\Phi_*\big(\sum_i s_* c^i \otimes (a_0^i \odot a)\big) \\
&= \psi_*\Phi_*(s_* \otimes id)\big(\sum_i c^i \otimes (a_0^i \odot a)\big) = \pi_{2*}\big(\sum_i c^i \otimes (a_0^i \odot a)\big) \\
&= \sum_i c^i(\mathbb{1}_{\mathscr{C}}) a_0^i \odot a = \psi_* b \odot a,
\end{aligned}$$

since $\psi_* b = \pi_{2*}(\sum_i c^i \otimes a_0^i) = \sum_i c^i(\mathbb{1}_{\mathscr{C}}) a_0^i$. ■

By Lemma 6.4 and Proposition 6.5, if we set $U = X$, it is straightforward that the condition of global triviality of a principal bundle remains valid in the graded setting.

6.6 Corollary-Theorem. *A graded principal bundle* $(Y, \mathscr{B})$ *is globally isomorphic to the product* $(X, \mathscr{C}) \times (G, \mathscr{A})$ *if and only if it admits a global section* $s\colon (X, \mathscr{C}) \to (Y, \mathscr{B})$.

We observe here that Lemma 6.4 and Proposition 6.5 remain valid if we replace the graded principal bundle $(Y, \mathscr{B})$ by a graded manifold $(Y, \mathscr{B})$ on which the graded Lie group $(G, \mathscr{A})$ acts freely to the right in such a way that the quotient $(X, \mathscr{C}) = (Y/G, \mathscr{B}/\mathscr{A})$ is a graded manifold, and the projection π: $(Y, \mathscr{B}) \to (X, \mathscr{C})$ is a submersion. In that case, one can construct via Lemma 6.4 and Proposition 6.5 the local trivializations of Definition 5.1. In other words:

6.7 Theorem. *A graded principal bundle is a graded manifold $(Y, \mathscr{B})$ together with a free right action Φ: $(Y, \mathscr{B}) \times (G, \mathscr{A}) \to (Y, \mathscr{B})$ of a graded Lie group $(G, \mathscr{A})$ such that:*

1. *the quotient $(X, \mathscr{C}) = (Y/G, \mathscr{B}/\mathscr{A})$ is a graded manifold*
2. *the projection π: $(Y, \mathscr{B}) \to (X, \mathscr{C})$ is a submersion.*

As an immediate application, we examine if the principal bundles formed by Lie groups and closed Lie subgroups possess graded analogs.

6.8 Example. Consider a graded Lie group $(G, \mathscr{A})$ and a closed graded Lie subgroup $(H, \mathscr{D})$ of $(G, \mathscr{A})$. The natural right action Φ: $(G, \mathscr{A}) \times (H, \mathscr{D}) \to (G, \mathscr{A})$ is given by $\Phi^* = (id \otimes i^*) \circ \Delta_{\mathscr{A}}$, where i: $(H, \mathscr{D}) \to (G, \mathscr{A})$ is the inclusion. Furthermore, we know that the quotient $(G/H, \mathscr{A}/\mathscr{D})$ is a graded manifold and the projection $(G, \mathscr{A}) \to (G/H, \mathscr{A}/\mathscr{D})$ is a submersion, [**15**]. Now if $g \in G$, the morphism Φ_{g*}: $\mathscr{D}(H)^\circ \to \mathscr{A}(G)^\circ$ is given by $\Phi_{g*}(d) = \delta_g \odot d = L_{g*}(d)$, $\forall d \in \mathscr{D}(H)^\circ$. As a result, Φ_{g*} is injective, so the action Φ is free and Theorem 6.7 holds: $(G, \mathscr{A})$ is a graded principal bundle over $(G/H, \mathscr{A}/\mathscr{D})$ with typical fibre $(H, \mathscr{D})$.

7. Lie superalgebra-valued graded differential forms

Let $(Y, \mathscr{B})$ be a graded manifold and $\mathfrak{g}$ a Lie superalgebra. We call $\mathfrak{g}$-valued graded differential form on $(Y, \mathscr{B})$, an element of $\Omega(Y, \mathscr{B}) \otimes \mathfrak{g}$. It is clear that the set $\Omega(Y, \mathscr{B}, \mathfrak{g}) = \Omega(Y, \mathscr{B}) \otimes \mathfrak{g}$ of these forms constitutes a $(\mathbf{Z} \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2)$-graded vector space; however, it is more convenient to introduce a $(\mathbf{Z} \oplus \mathbf{Z}_2)$-grading as follows: if $\alpha \in \Omega(Y, \mathscr{B}, \mathfrak{g})$ and $\deg(\alpha) = (i_\alpha, j_\alpha, k_\alpha)$ is its $(\mathbf{Z} \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2)$-degree, then we set $|\alpha| = (i_\alpha, j_\alpha + k_\alpha) \in \mathbf{Z} \oplus \mathbf{Z}_2$. If $\{e_i\}$ is a basis of the Lie

superalgebra $\mathfrak{g}$ and $\alpha, \beta \in \Omega(Y, \mathscr{B}, \mathfrak{g})$, then we may write $\alpha = \sum_i \alpha^i \otimes e_i$ and $\beta = \sum_i \beta^i \otimes e_i$, for α^i, $\beta^i \in \Omega(Y, \mathscr{B})$. In the case where α and β are homogeneous with $\deg(\alpha) = (i_\alpha, j_\alpha, k_\alpha)$, $\deg(\beta) = (i_\beta, j_\beta, k_\beta)$, we define the $\mathfrak{g}$-valued graded differential form $[\alpha, \beta]$ of degree $\deg([\alpha, \beta]) = (i_\alpha + i_\beta, j_\alpha + j_\beta, k_\alpha + k_\beta)$ as:

$$[\alpha, \beta] = \sum_{i,j} (-1)^{j_\beta k_\alpha} \alpha^i \beta^j \otimes [e_i^\alpha, e_j^\beta], \tag{7.1}$$

if $\alpha = \sum_i \alpha^i \otimes e_i^\alpha$ with $|\alpha^i| = (i_\alpha, j_\alpha)$ and $e_i^\alpha = k_\alpha$; similarly for β. We extend this definition to non-homogeneous elements by linearity. Clearly, equation (7.1) gives the same result for every basis of the Lie superalgebra $\mathfrak{g}$. Thus, we have a bilinear map $[-,-]$: $\Omega^i(Y, \mathscr{B})_j \otimes \mathfrak{g}_k \times \Omega^{i'}(Y, \mathscr{B})_{j'} \otimes \mathfrak{g}_{k'} \to \Omega^{i+i'}(Y, \mathscr{B})_{j+j'} \otimes \mathfrak{g}_{k+k'}$ with the following properties:

7.1 Proposition. *If $\alpha, \beta, \gamma \in \Omega(Y, \mathscr{B}, \mathfrak{g})$ are homogeneous, then we have:*

1. $[\alpha, \beta] = -(-1)^{|\alpha||\beta|}[\beta, \alpha]$
2. $\mathfrak{S}(-1)^{|\alpha||\gamma|}\big[[\alpha, \beta], \gamma\big] = 0.$

In the previous relations, we have set $|\alpha||\beta| = i_\alpha i_\beta + (j_\alpha + k_\alpha)(j_\beta + k_\beta)$ and $\mathfrak{S}$ means the cyclic sum on the argument which follows.

Proof. Routine calculations using relation (7.1). ■

We realize that the space $\Omega(Y, \mathscr{B}, \mathfrak{g})$ possesses the structure of a $(\mathbf{Z} \oplus \mathbf{Z}_2)$-graded Lie algebra, inherited from the the Lie superalgebra structure of $\mathfrak{g}$.

The action now of elements of $\Omega(Y, \mathscr{B}, \mathfrak{g})$ on derivations can be seen as follows: if $\alpha = \sum_i \alpha^i \otimes e_i \in \Omega(Y, \mathscr{B}, \mathfrak{g})$ is an r-form and $\xi_1, \ldots, \xi_r \in \mathfrak{Der}\mathscr{B}(Y)$, we set $(\xi_1, \ldots, \xi_r|\alpha) = \sum_i (\xi_1, \ldots, \xi_r|\alpha^i) \otimes e_i$. Accordingly, one can extend the exterior differential d to a differential on $\mathfrak{g}$-valued graded differential forms, also noted by d in the following manner: if $\alpha = \sum_i \alpha^i \otimes e_i$, then we set $d\alpha = \sum_i d\alpha^i \otimes e_i$. The exterior differential d: $\Omega(Y, \mathscr{B}, \mathfrak{g}) \to \Omega(Y, \mathscr{B}, \mathfrak{g})$ defined previously, is a derivation of degree $|d| = (1, 0)$. By straightforward verification, we find that, if α, β are $\mathfrak{g}$-valued graded differential forms and α is homogeneous, then $d[\alpha, \beta] = [d\alpha, \beta] + (-1)^{|\alpha||d|}[\alpha, d\beta]$.

In a similar way, one can extend the pull-back of graded differential forms under a morphism of graded manifolds σ: $(Y, \mathscr{B}) \to (Z, \mathscr{Y})$ to a linear map

$\sigma^*\colon \Omega(Z,\mathscr{Y},\mathfrak{g}) \to \Omega(Y,\mathscr{B},\mathfrak{g})$ which commutes with the exterior differential, that is, $d\circ\sigma^* = \sigma^*\circ d$, and preserves the bracket $[-,-]$: $\sigma^*[\alpha,\beta] = [\sigma^*\alpha,\sigma^*\beta]$. We have analogous generalizations for the Lie derivative. The following properties of the bracket and the Lie derivative on $\mathfrak{g}$-valued graded differential forms will be useful; the proof proceeds by a straightforward calculation with graded differential forms and Lie superalgebra elements, and is left as an exercise for the reader.

7.2 Proposition. *If $\alpha,\beta\in\Omega(Y,\mathscr{B},\mathfrak{g})$ and $\xi\in\mathfrak{Der}\mathscr{B}(Y)$ are homogeneous, then*

1. $\boldsymbol{L}_\xi[\alpha,\beta] = [\boldsymbol{L}_\xi\alpha,\beta] + (-1)^{|\boldsymbol{L}_\xi||\alpha|}[\alpha,\boldsymbol{L}_\xi\beta]$
2. $\big(id\otimes ad(v)\big)[\alpha,\beta] = \big[(id\otimes ad(v))\alpha,\beta\big] + (-1)^{|v||\alpha|}\big[\alpha,(id\otimes ad(v))\beta\big]$, $\forall v\in\mathfrak{g}$.

Finally, one can restate Proposition 2.4 for the case of $\mathfrak{g}$-valued graded differential forms. For example, if σ is an isomorphism of graded manifolds, relation (2.12) becomes:

$$(\xi_1,\dots,\xi_r|\sigma^*\alpha) = (\sigma^*\otimes id)(\sigma_*\xi_1,\dots,\sigma_*\xi_r|\alpha). \tag{7.2}$$

One can also define multilinear forms on the tangent spaces of a graded manifold taking its values in the Lie superalgebra $\mathfrak{g}$, using a simple modification of (2.11).

8. Graded connections

We have seen that on each graded principal bundle there always exists a natural distribution induced by the action of the structure group which is equal to the sheaf of vertical derivations. The choice of a connection is essentially the choice of a complementary distribution. More precisely:

8.1 Definition. *Let $(Y,\mathscr{B})$ be a graded principal bundle with structure group $(G,\mathscr{A})$, over the graded manifold $(X,\mathscr{C})$. A graded connection on $(Y,\mathscr{B})$ is a regular distribution $\mathscr{H}\subset\mathfrak{Der}\mathscr{B}$ of dimension $\dim\mathscr{H} = \dim(X,\mathscr{C})$ such that:*

1. $\mathscr{H}\oplus\mathscr{V}\!\mathit{er}(\pi_*,\mathscr{B}) = \mathfrak{Der}\mathscr{B}$, $\pi\colon (Y,\mathscr{B})\to(X,\mathscr{C})$ *is the projection*
2. $\mathscr{H}$ *is $(G,\mathscr{A})$-invariant.*

Let us explain the second statement in the previous definition: $\mathscr{H}$ will be called $(G, \mathscr{A})$-invariant if, for each open $U \subset X$ and $D \in \mathscr{H}(\pi_*^{-1}(U))$, the derivations $\Phi_g^* D$ and $[(\Phi^*)_a, D] = \boldsymbol{L}_{(\Phi^*)_a} D$ belong also to $\mathscr{H}(\pi_*^{-1}(U))$, $\forall g \in G$, $\forall a \in \mathfrak{g}$. In order to put these condition in a more compact form, we introduce the following notation

$$(\Phi^*)_a D = \begin{cases} \Phi_g^* D, & \text{if} \quad a = \delta_g, g \in G \\ \boldsymbol{L}_{(\Phi^*)_a} D, & \text{if} \quad a \in \mathfrak{g}. \end{cases}$$

Then, $\mathscr{H}$ will be $(G, \mathscr{A})$-invariant if $(\Phi^*)_a D \in \mathscr{H}(\pi_*^{-1}(U))$, for each element a group-like or primitive with respect to δ_e. We can now reformulate this notion in terms of $\mathfrak{g}$-valued graded differential forms.

Given the graded connection $\mathscr{H}$ on $(Y, \mathscr{B})$, we have: $\mathscr{H}(Y) \oplus \mathscr{E}(Y) = \mathfrak{Der}\mathscr{B}(Y)$, $\mathscr{E} = \mathscr{Ver}(\pi_*, \mathscr{B})$ (see Theorem 6.1), and each derivation $\xi \in \mathfrak{Der}\mathscr{B}(Y)$ decomposes as $\xi = \xi^H + \sum_i f^i (\Phi^*)_{e_i}$, where $\xi^H \in \mathscr{H}(Y)$, $\{e_i\}$ is a basis of $\mathfrak{g}$ and $f^i \in \mathscr{B}(Y)$. Then, we define a 1-form $\boldsymbol{\omega} \in \Omega^1(Y, \mathscr{B}, \mathfrak{g})$ setting $(\xi | \boldsymbol{\omega}) = \sum_i f^i \otimes e_i$. Let us now calculate $(\Phi^*)_a \boldsymbol{\omega}$, where by definition

$$(\Phi^*)_a \boldsymbol{\omega} = \begin{cases} \Phi_g^* \boldsymbol{\omega}, & \text{if} \quad a = \delta_g, g \in G \\ \boldsymbol{L}_{(\Phi^*)_a} \boldsymbol{\omega}, & \text{if} \quad a \in \mathfrak{g}. \end{cases}$$

Consider first the case $a = \delta_g$. Then, by equation (7.2) we take

$$(\xi | \Phi_g^* \boldsymbol{\omega}) = (\Phi_g^* \otimes id)(\Phi_{g*}\big(\sum_i f^i (\Phi^*)_{e_i}\big) | \boldsymbol{\omega})$$

and using the fact that $\Phi_{g*}(\Phi^*)_{e_i} = (\Phi^*)_{AD_{g^{-1}*}(e_i)}$, we obtain:

$$(\xi | \Phi_g^* \boldsymbol{\omega}) = (\Phi_g^* \otimes id)\big(\sum_i \Phi_{g^{-1}}^* f^i \otimes AD_{g^{-1}*}(e_i)\big) = (id \otimes AD_{g^{-1}*})(\xi | \boldsymbol{\omega}).$$

or

$$\Phi_g^* \boldsymbol{\omega} = (id \otimes AD_{g^{-1}*}) \circ \boldsymbol{\omega}. \tag{8.1}$$

If now $a \in \mathfrak{g}$, we have:

$$(\xi | \boldsymbol{L}_{(\Phi^*)_a} \boldsymbol{\omega}) = (-1)^{|a||\xi|} \Big[\big((\Phi^*)_a \otimes id\big)(\xi | \boldsymbol{\omega}) - \big([(\Phi^*)_a, \xi] | \boldsymbol{\omega}\big) \Big] \tag{8.2}$$

and it is sufficient to examine two cases:

1. $\xi = \xi^H$ (horizontal derivation): it is then immediate that $(\xi|\boldsymbol{L}_{(\Phi^*)_a}\boldsymbol{\omega}) = 0$
2. $\xi = (\Phi^*)_b, b \in \mathfrak{g}$ (vertical derivation):

$$(\xi|\boldsymbol{L}_{(\Phi^*)_a}\boldsymbol{\omega}) = (-1)^{|a||b|+1}\big((\Phi^*)_{[a,b]}|\boldsymbol{\omega}\big) = -\big((\Phi^*)_b|(id \otimes AD_{*a})\boldsymbol{\omega}\big)$$

where we have defined $\big(\xi|(id \otimes AD_{*a})\big) \circ \boldsymbol{\omega}\big) = (-1)^{|a||\xi|}(id \otimes AD_{*a})(\xi|\boldsymbol{\omega})$.
We may thus write:

$$\boldsymbol{L}_{(\Phi^*)_a}\boldsymbol{\omega} = -(id \otimes AD_{*a}) \circ \boldsymbol{\omega}. \tag{8.3}$$

Summarizing the previous results on the $\mathfrak{g}$-valued graded differential form $\boldsymbol{\omega}$, we have:

1. $(\sum_i f^i(\Phi^*)_{e_i}|\boldsymbol{\omega}) = \sum_i f^i \otimes e_i$ and $|\boldsymbol{\omega}| = (1, 0)$
2. $(\Phi^*)_a\boldsymbol{\omega} = (id \otimes AD_{*s^\circ_{\mathscr{A}}(a)}) \circ \boldsymbol{\omega}$, for each a group-like or primitive with respect to δ_e.

Conversely now, if $\boldsymbol{\omega}$ is a $\mathfrak{g}$-valued graded differential form with the previous two properties, then $\ker\boldsymbol{\omega} = \mathscr{H}$ is a regular distribution on $(Y, \mathscr{B})$ such that $\mathscr{H} \oplus \mathscr{V}\!\mathit{er}(\pi_*, \mathscr{B}) = \mathfrak{Der}\mathscr{B}$. Consider $D \in \mathscr{H}(V)$, $V = \pi_*^{-1}(U)$; if $a = \delta_g$ and $\boldsymbol{\omega}_V$ means the pull-back of $\boldsymbol{\omega}$ under the inclusion $(V, \mathscr{B}|_V) \hookrightarrow (Y, \mathscr{B})$, we take: $(D|\Phi_g^*\boldsymbol{\omega}_V) = (\Phi_g^* \otimes id)(\Phi_{g*}D|\boldsymbol{\omega}_V) = (id \otimes AD_{g^{-1}*})(D|\boldsymbol{\omega}_V) = 0$; thus, $\Phi_{g*}D \in \mathscr{H}(V)$. Replacing g by g^{-1}, this gives $\Phi_g^*D \in \mathscr{H}(V)$. On the other hand, if $a \in \mathfrak{g}$, we find: $(D|\boldsymbol{L}_{(\Phi^*)_a}\boldsymbol{\omega}_V) = -(-1)^{|a||D|}([(\Phi^*)_a, D]|\boldsymbol{\omega}_V) = 0$. This means that $\boldsymbol{L}_{(\Phi^*)_a}D$ belongs also to $\mathscr{H}(V)$. In summary: $(\Phi^*)_aD \in \mathscr{H}(V)$. We have thus proved the proposition.

8.2 Proposition. *The graded connection $\mathscr{H}$ of Definition 8.1 is described equivalently by a $\mathfrak{g}$-valued graded differential 1-form $\boldsymbol{\omega} \in \Omega^1(Y, \mathscr{B}, \mathfrak{g})$ of total $\mathbf{Z}_2$ degree zero such that:*

1. $(\sum_i f^i(\Phi^*)_{e_i}|\boldsymbol{\omega}) = \sum_i f^i \otimes e_i$,
2. $(\Phi^*)_a\boldsymbol{\omega} = (id \otimes AD_{*s^\circ_{\mathscr{A}}(a)}) \circ \boldsymbol{\omega}$, *for each element a group-like or primitive with respect to δ_e.*

We call $\boldsymbol{\omega}$ graded connection form.

Graded principal bundles are by definition locally isomorphic to products of open graded submanifolds of the base space by the structure graded Lie group. So, it is quite natural to ask how one can construct graded connections on the trivial graded principal bundle of Example 5.3.

8.3 Example. Let $(Y, \mathscr{B}) = (X, \mathscr{C}) \times (G, \mathscr{A})$ be as in Example 5.3. The right action Φ of $(G, \mathscr{A})$ on $(Y, \mathscr{B})$ is such that $\Phi^* = id \otimes \Delta_{\mathscr{A}}$. Consider now a $\mathfrak{g}$-valued 1-form $\beta \in \Omega^1(X, \mathscr{C}, \mathfrak{g})$ and a basis $\{e_k\}$ of $\mathfrak{g}$. One can write $\beta = \sum_k \beta^k \otimes e_k$, $\beta^k \in \Omega^1(Y, \mathscr{B})$ with $|\beta| = (1,0)$. If $\xi \in \mathfrak{Der}\mathscr{C}(X)$ and $\eta \in \mathfrak{Der}\mathscr{A}(G)$, we define a $\mathfrak{g}$-valued 1-form $\boldsymbol{\omega} \in \Omega^1(Y, \mathscr{B}, \mathfrak{g})$ as:

$$(\xi \otimes \mathbb{1}_{\mathscr{A}} + \mathbb{1}_{\mathscr{C}} \otimes \eta | \boldsymbol{\omega}) = \sum_k (\xi|\beta^k) \otimes ((L^*)_{e_k}|\theta) + \mathbb{1}_{\mathscr{C}} \otimes (\eta|\theta). \tag{8.4}$$

In the previous relation L: $(G, \mathscr{A}) \times (G, \mathscr{A}) \to (G, \mathscr{A})$ is the left action of $(G, \mathscr{A})$ on itself, $L^* = \Delta_{\mathscr{A}}$ and $\theta \in \Omega^1(G, \mathscr{A}, \mathfrak{g})$ the graded Maurer-Cartan form on $(G, \mathscr{A})$ defined as $((R^*)_a|\theta) = \mathbb{1}_{\mathscr{A}} \otimes a$, $\forall a \in \mathfrak{g}$, where R: $(G, \mathscr{A}) \times (G, \mathscr{A}) \to (G, \mathscr{A})$ is the right action of $(G, \mathscr{A})$ on itself, $R^* = \Delta_{\mathscr{A}}$. Recall that the derivations $(L^*)_{e_k}$ and $(R^*)_a$ are given by Theorem 3.9. We shall check now if the form $\boldsymbol{\omega}$ in (8.4) has the properties of a graded connection form.

(1) If $a \in \mathfrak{g}$, then $(\Phi^*)_a = id \otimes (R^*)_a$ and $((\Phi^*)_a|\boldsymbol{\omega}) = \mathbb{1}_{\mathscr{C}} \otimes ((R^*)_a|\theta) = \mathbb{1}_{\mathscr{B}} \otimes a$. Further, $|\boldsymbol{\omega}| = (1,0)$ because $|\beta| = (1,0)$ and $|\theta| = (1,0)$.

(2) Consider now an element $g \in G$; we shall calculate $\Phi_g^* \boldsymbol{\omega}$. To this end, we use formula (7.2), as well as the fact that each graded Lie group is parallelizable (Theorem 3.14), so it is sufficient to take $\eta = (R^*)_a$, $a \in \mathfrak{g}$. Thus, if we set $D = \xi \otimes \mathbb{1}_{\mathscr{A}} + \mathbb{1}_{\mathscr{C}} \otimes (R^*)_a$ and $a' = AD_{g^{-1}*}(a)$, we find

$$\begin{aligned}(D|\Phi_g^* \boldsymbol{\omega}) &= (\Phi_g^* \otimes id)(\xi \otimes \mathbb{1}_{\mathscr{A}} + \mathbb{1}_{\mathscr{C}} \otimes (R^*)_{a'} | \boldsymbol{\omega}) \\ &= \sum_k (\xi|\beta^k) \otimes (R_g^* \otimes id)((L^*)_{e_k}|\theta) + \mathbb{1}_{\mathscr{B}} \otimes AD_{g^{-1}*}(a) \\ &= \sum_k (\xi|\beta^k) \otimes (id \otimes AD_{g^{-1}*})((L^*)_{e_k}|\theta) + \mathbb{1}_{\mathscr{B}} \otimes AD_{g^{-1}*}(a) \\ &= (\xi \otimes \mathbb{1}_{\mathscr{A}} + \mathbb{1}_{\mathscr{C}} \otimes (R^*)_a | (id \otimes AD_{g^{-1}*}) \circ \boldsymbol{\omega}).\end{aligned}$$

Note that in the previous calculation we used that $R_g^* \theta = (id \otimes AD_{g^{-1}*}) \circ \theta$ and $R_{g*}(L^*)_{e_k} = (L^*)_{e_k}$, which can be verified straightforwardly.

(3) Let finally $a \in \mathfrak{g}$; we calculate the Lie derivative $\boldsymbol{L}_{(\Phi^*)_a}\boldsymbol{\omega} = (\Phi^*)_a\boldsymbol{\omega}$. One can use formula (8.2), which is valid for $\boldsymbol{\omega}$ because $((\Phi^*)_a|\boldsymbol{\omega}) = \mathbb{1}_{\mathscr{B}} \otimes a$. If we set now $D = \xi \otimes \mathbb{1}_{\mathscr{A}} + \mathbb{1}_{\mathscr{C}} \otimes (R^*)_b$ and $x = |a||\beta^k|$, we have:

$$\begin{aligned}(D|(\Phi^*)_a\boldsymbol{\omega}) &= (-1)^{|a||\xi|}((\Phi^*)_a \otimes id)(\xi \otimes \mathbb{1}_{\mathscr{A}}|\boldsymbol{\omega}) - (-1)^{|a||b|}(\mathbb{1}_{\mathscr{C}} \otimes (R^*)_{[a,b]}|\boldsymbol{\omega}) \\ &= (-1)^x(\xi|\beta^k) \otimes ((R^*)_a \otimes id)((L^*)_{e_k}|\theta) - (-1)^{|a||b|}\mathbb{1}_{\mathscr{B}} \otimes [a,b] \\ &= -(-1)^x(\xi|\beta^k) \otimes (id \otimes AD_{*a})((L^*)_{e_k}|\theta) - (-1)^{|a||b|}\mathbb{1}_{\mathscr{B}} \otimes [a,b] \\ &= -(\xi \otimes \mathbb{1}_{\mathscr{A}}|(id \otimes AD_{*a}) \circ \boldsymbol{\omega}) - (\mathbb{1}_{\mathscr{C}} \otimes (R^*)_b|(id \otimes AD_{*a}) \circ \boldsymbol{\omega})\end{aligned}$$

which implies that $(\Phi^*)_a\boldsymbol{\omega} = -(id \otimes AD_{*a}) \circ \boldsymbol{\omega}$. In the previous calculation, summation over the repeated index k is understood. Note also that we used the property $(R^*)_a\theta = -(id \otimes AD_{*a}) \circ \theta$ of the graded Maurer-Cartan form.

8.4 Remark. *Let s: $(X,\mathscr{C}) \to (Y,\mathscr{B})$ be a graded section for the previous example. Then we have a morphism s^*: $\mathscr{C}(X) \otimes \mathscr{A}(G) \to \mathscr{C}(X)$ of graded commutative algebras and therefore, a morphism σ^*: $\mathscr{A}(G) \to \mathscr{C}(X)$ given by $\sigma^*(f_{\mathscr{A}}) = s^*(\mathbb{1}_{\mathscr{C}} \otimes f_{\mathscr{A}})$. This defines a morphism of graded manifolds σ: $(X,\mathscr{C}) \to (G,\mathscr{A})$. Let now $\xi \in \mathfrak{Der}\mathscr{C}(X)$ and $\eta \in \mathfrak{Der}\mathscr{A}(G)$ be two σ-related derivations; then the derivations $\xi' = \xi \otimes \mathbb{1}_{\mathscr{A}} + \mathbb{1}_{\mathscr{C}} \otimes \eta$ and ξ are s-related and one can use Proposition 2.4 in order to calculate the pull-back $s^*\boldsymbol{\omega} \in \Omega^1(X,\mathscr{C},\mathfrak{g})$. The result is:*

$$s^*\boldsymbol{\omega} = \textstyle\sum_k \beta^k(\sigma^* \otimes id)((L^*)_{e_k}|\theta) + \sigma^*\theta,$$

as one easily finds.

One can use Example 8.3 in order to construct graded connections on general graded principal bundles. Indeed, let $(Y,\mathscr{B})$ be such a bundle with base space $(X,\mathscr{C})$ and structure group $(G,\mathscr{A})$, $\{U_i\}_{i \in \Lambda}$ a locally finite open covering of X and $\{f_i\}$ a graded partition of unity subordinate to $\{U_i\}$: $f_i \in \mathscr{C}(X)_0$, $\operatorname{supp} f_i \subset U_i$ and $\sum_i f_i = \mathbb{1}_{\mathscr{C}}$, [**15**]. Let also $V_i = \pi_*^{-1}(U_i)$ and $\boldsymbol{\omega}_i$ be the graded connection 1-forms that one can construct on $(V_i, \mathscr{B}|_{V_i}) \cong (U_i, \mathscr{C}|_{U_i}) \times (G,\mathscr{A})$ as in the previous example (see also Definition 5.1). If now $D \in \mathfrak{Der}\mathscr{B}(Y)$, then we define $(D|\boldsymbol{\omega}_i) \in \mathscr{B}(V_i) \otimes \mathfrak{g}$ as $(D|\boldsymbol{\omega}_i) = (D_i|\boldsymbol{\omega}_i)$, $D_i = D|_{V_i}$. Then we have also $(D|\boldsymbol{\omega}_i) \cdot (\pi^* f_i|_{V_i}) \in \mathscr{B}(V_i) \otimes \mathfrak{g}$ and so, there exists an element of $\mathscr{B}(Y) \otimes \mathfrak{g}$, which we

denote by $(D|\boldsymbol{\omega}_i)\pi^* f_i$, such that $[(D|\boldsymbol{\omega}_i)\pi^* f_i]|_{V_i} = (D|\boldsymbol{\omega}_i)(\pi^* f_i|_{V_i})$ and whose support is a subset of V_i.

If now $\{\tilde{U}_k\}$ is another open covering of X, then setting $W_k = \pi_*^{-1}(\tilde{U}_k)$ we have an open covering of Y and for k fixed, $W_k \cap V_i$ is non-empty only for finitely many of the V_i. Taking the restrictions $[(D|\boldsymbol{\omega}_i)\pi^* f_i]|_{W_k} \in \mathscr{B}(W_k) \otimes \mathfrak{g}$, only finite many terms will be non-zero as i runs over Λ. Thus, the sum $\sum_i [(D|\boldsymbol{\omega}_i)\pi^* f_i]|_{W_k}$ is finite and well-defined as an element of $\mathscr{B}(W_k) \otimes \mathfrak{g}$. Furthermore, the restrictions of such elements to the intersections $W_k \cap W_\ell$ coincide and this means, by the sheaf properties of $\mathscr{B}$, that there exists a unique element of $\mathscr{B}(Y) \otimes \mathfrak{g}$, whose restriction to W_k gives the previous element of $\mathscr{B}(W_k) \otimes \mathfrak{g}$. We denote this unique element by $(D|\boldsymbol{\omega})$ and by its linearity on D it determines a $\mathfrak{g}$-valued graded 1-form $\boldsymbol{\omega}$. The form $\boldsymbol{\omega}$ is a graded connection form; this results without difficulty from the properties $\Phi_g^* \pi^* f_i = \pi^* f_i$ and $(\Phi^*)_a \pi^* f_i = 0$, $g \in G$, $a \in \mathfrak{g}$, as well as from the fact that for each $i \in \Lambda$, $\boldsymbol{\omega}_i$ is a graded connection. We have thus proved the following:

8.5 Existence Theorem for graded connections. *On each graded principal bundle there exists an infinity of graded connections.*

9. Graded curvature

In ordinary differential geometry, one can define in a canonical way, for each connection, a Lie superalgebra-valued 2-form, the curvature form. In this section, we will define the curvature in the graded setting, using the notion of graded connection, previously developed.

Let then $\boldsymbol{\omega} \in \Omega^1(Y, \mathscr{B}, \mathfrak{g})$ be a graded connexion form on the graded principal bundle $(Y, \mathscr{B})$. Fix a basis of the Lie $\{e_k\}$ of the Lie superalgebra $\mathfrak{g}$; then for each derivation $\xi \in \mathfrak{Der}\mathscr{B}(Y)$ one can find $\xi^H \in \mathscr{H}(Y)$ and $f^k \in \mathscr{B}(Y)$ such that $\xi = \xi^H + \sum_i f^k (\Phi^*)_{e_k}$. We have thus a canonical projection $\mathfrak{Der}\mathscr{B}(Y) \to \mathscr{H}(Y)$, $\xi \mapsto \xi^H$; we call ξ^H horizontal part of ξ. Clearly, this mechanism of taking the horizontal part of a derivation can be applied in the same way if instead of Y we put an open $V \subset Y$.

Consider now a $\mathfrak{g}$-valued graded differential form $\phi \in \Omega^r(Y, \mathscr{B}, \mathfrak{g})$ and let ϕ^H be defined as $(\xi_1, \ldots, \xi_r | \phi^H) = (\xi_1^H, \ldots, \xi_r^H | \phi)$, $\xi_i \in \mathfrak{Der}\mathscr{B}(Y)$, $i = 1, \ldots, r$.

9.1 Definition. *The covariant exterior derivative of* $\phi \in \Omega^r(Y, \mathscr{B}, \mathfrak{g})$ *is the* $\mathfrak{g}$*-valued graded differential form* $D^{\boldsymbol{\omega}}\phi \in \Omega^{r+1}(Y, \mathscr{B}, \mathfrak{g})$ *defined as* $D^{\boldsymbol{\omega}}\phi = (d\phi)^H$. *The curvature of the graded connection* $\boldsymbol{\omega} \in \Omega^1(Y, \mathscr{B}, \mathfrak{g})$ *is the covariant exterior derivative* $D^{\boldsymbol{\omega}}\boldsymbol{\omega}$*; we use the notation* $F^{\boldsymbol{\omega}} = D^{\boldsymbol{\omega}}\boldsymbol{\omega}$.

We will next study in more detail the properties of $F^{\boldsymbol{\omega}}$. As a general observation, we may say that $F^{\boldsymbol{\omega}}$ has, formally, properties analogous to those of the ordinary curvature.

9.2 Theorem. *The graded curvature* $F^{\boldsymbol{\omega}}$ *is given by the relation*

$$F^{\boldsymbol{\omega}} = d\boldsymbol{\omega} + \frac{1}{2}[\boldsymbol{\omega}, \boldsymbol{\omega}]. \tag{9.1}$$

This is the graded structure equation.

Proof. It is sufficient to prove that for each ξ_1, $\xi_2 \in \mathfrak{Der}\mathscr{B}(Y)$, the following is true:

$$(\xi_1^H, \xi_2^H | d\boldsymbol{\omega}) = (\xi_1, \xi_2 | d\boldsymbol{\omega}) + \frac{1}{2}(\xi_1, \xi_2 | [\boldsymbol{\omega}, \boldsymbol{\omega}]). \tag{9.2}$$

As we have seen $|\boldsymbol{\omega}| = (1, 0)$; however, $\boldsymbol{\omega}$ is not a homogeneous element with respect to the $(\mathbf{Z} \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2)$-grading of $\Omega(Y, \mathscr{B}, \mathfrak{g})$. More precisely, $\boldsymbol{\omega} = \boldsymbol{\omega}_0 + \boldsymbol{\omega}_1$, where $\deg(\boldsymbol{\omega}_0) = (1, 0, 0)$, $\deg(\boldsymbol{\omega}_1) = (1, 1, 1)$ and if $\{e_i, e_j\}$ is a basis of $\mathfrak{g}$ with $|e_i| = 0$, $|e_j| = 1$, we may write: $\boldsymbol{\omega} = \sum_i \boldsymbol{\omega}^i \otimes e_i + \sum_j \boldsymbol{\omega}^j \otimes e_j$, with $\boldsymbol{\omega}^i \in \Omega^1(Y, \mathscr{B})_0$, $\boldsymbol{\omega}^j \in \Omega^1(Y, \mathscr{B})_1$. In particular, $\boldsymbol{\omega}^i$, $\boldsymbol{\omega}^j$ vanish on horizontal derivations and if we decompose $a \in \mathfrak{g}$ as $a = \sum_i a^i e_i + \sum_j a^j e_j$, we find immediately $((\Phi^*)_a | \boldsymbol{\omega}^i) = a^i \mathbb{1}_{\mathscr{B}}$, $((\Phi^*)_a | \boldsymbol{\omega}^j) = a^j \mathbb{1}_{\mathscr{B}}$. Let us now calculate the term $\frac{1}{2}(\xi_1, \xi_2 | [\boldsymbol{\omega}, \boldsymbol{\omega}])$.

$$\begin{aligned}
(\xi_1, \xi_2 | [\boldsymbol{\omega}, \boldsymbol{\omega}]) &= \left(\xi_1, \xi_2 \middle| \left[\textstyle\sum_i \boldsymbol{\omega}^i \otimes e_i + \sum_j \boldsymbol{\omega}^j \otimes e_j, \sum_p \boldsymbol{\omega}^p \otimes e_p + \sum_q \boldsymbol{\omega}^q \otimes e_q\right]\right) \\
&= \textstyle\sum_{i,p} \Big((\xi_1 | \boldsymbol{\omega}^i)(\xi_2 | \boldsymbol{\omega}^p) + (-1)^{1+|\xi_1||\xi_2|}(\xi_2 | \boldsymbol{\omega}^i)(\xi_1 | \boldsymbol{\omega}^p) \Big) \otimes [e_i, e_p] \\
&+ \textstyle\sum_{i,q} \Big((\xi_1 | \boldsymbol{\omega}^i)(\xi_2 | \boldsymbol{\omega}^q) + (-1)^{1+|\xi_1||\xi_2|}(\xi_2 | \boldsymbol{\omega}^i)(\xi_1 | \boldsymbol{\omega}^q) \Big) \otimes [e_i, e_q] \\
&+ \textstyle\sum_{j,p} \Big((-1)^{|\xi_2|}(\xi_1 | \boldsymbol{\omega}^j)(\xi_2 | \boldsymbol{\omega}^p) + (-1)^x (\xi_2 | \boldsymbol{\omega}^j)(\xi_1 | \boldsymbol{\omega}^p) \Big) \otimes [e_j, e_p] \\
&- \textstyle\sum_{j,q} \Big((-1)^{|\xi_2|}(\xi_1 | \boldsymbol{\omega}^j)(\xi_2 | \boldsymbol{\omega}^q) + (-1)^x (\xi_2 | \boldsymbol{\omega}^j)(\xi_1 | \boldsymbol{\omega}^q) \Big) \otimes [e_j, e_q].
\end{aligned} \tag{9.3}$$

In the previous calculation, the indices i, p label the even elements while j, q the odd ones. We also used the fact that if $\beta_1, \beta_2 \in \Omega^1(Y, \mathscr{B})$, then $(\xi_1, \xi_2|\beta_1\beta_2) = (-1)^{|\xi_2||\beta_1|}(\xi_1|\beta_1)(\xi_2|\beta_2) + (-1)^{1+|\xi_1||\xi_2|+|\xi_1||\beta_1|}(\xi_2|\beta_1)(\xi_1|\beta_2)$, see relation 4.1.9 of [**15**], and we have set $x = 1+|\xi_1||\xi_2|+|\xi_1|$. On the other hand, the term $(\xi_1, \xi_2|d\boldsymbol{\omega})$ is calculated via $(\xi_1, \xi_2|d\boldsymbol{\omega}) = (\xi_1 \otimes id)(\xi_2|\boldsymbol{\omega}) - (-1)^{|\xi_1||\xi_2|}(\xi_2 \otimes id)(\xi_1|\boldsymbol{\omega}) - ([\xi_1, \xi_2]|\boldsymbol{\omega})$, which is an immediate generalization of 4.3.10 of [**15**]. We distinguish now the following cases:

1. ξ_1, ξ_2: horizontal $\Rightarrow \xi_1 = \xi_1^H, \xi_2 = \xi_2^H$. The graded structure equation holds, since $\frac{1}{2}(\xi_1, \xi_2|[\boldsymbol{\omega}, \boldsymbol{\omega}]) = 0$ and $(\xi_i|\boldsymbol{\omega}) = 0$.
2. ξ_1: horizontal, $\xi_2 = (\Phi^*)_a$, $a \in \mathfrak{g}$. The left-hand side of (9.2) is zero because $\xi_2^H = 0$. The right-hand side of the same equation reads: $(\xi_1, (\Phi^*)_a|d\boldsymbol{\omega}) = (\xi_1 \otimes id)((\Phi^*)_a|\boldsymbol{\omega}) - ([\xi_1, (\Phi^*)_a]|\boldsymbol{\omega}) = 0$, because by the definition of the graded connexion, if ξ is horizontal, $[\xi, (\Phi^*)_a]$ is horizontal too. Furthermore, it is clear that in this case, (9.3) gives $(\xi_1, \xi_2|[\boldsymbol{\omega}, \boldsymbol{\omega}]) = 0$.
3. $\xi_1 = (\Phi^*)_a$, $\xi_2 = (\Phi^*)_b$, $a, b \in \mathfrak{g}$. Clearly, the left-hand side of (9.2) is zero. Examining now the cases $|a| = |b| = 0$, $|a| = |b| = 1$ and $|a| = 1, |b| = 0$, we find always that the right-hand side is also zero. ■

By the graded structure equation, it is clear that $\boldsymbol{\omega}$ and $F^{\boldsymbol{\omega}}$ have the same $\mathbf{Z}_2$ total degree: $|F^{\boldsymbol{\omega}}| = (2, 0)$; therefore, $F^{\boldsymbol{\omega}} = F_0^{\boldsymbol{\omega}} + F_1^{\boldsymbol{\omega}}$ with $F_0^{\boldsymbol{\omega}} \in \Omega^2(Y, \mathscr{B})_0 \otimes \mathfrak{g}_0$, $F_1^{\boldsymbol{\omega}} \in \Omega^2(Y, \mathscr{B})_1 \otimes \mathfrak{g}_1$.

Bianchi's identity is a well-known property satisfied by the curvature in differential geometry. Using the generalization of the covariant derivative in the graded setting and the previous theorem, we may establish an analogous property in the context of graded manifolds.

9.3 Proposition. (Bianchi's Identity) $D^{\boldsymbol{\omega}} F^{\boldsymbol{\omega}} = 0$.

Proof. Let us first calculate the differential $dF^{\boldsymbol{\omega}}$. Using the graded structure equation and the fact that $\big[[\boldsymbol{\omega}, \boldsymbol{\omega}], \boldsymbol{\omega}\big] = 0$ (Jacobi identity), we find easily:

$$dF^{\boldsymbol{\omega}} = \frac{1}{2} d[\boldsymbol{\omega}, \boldsymbol{\omega}] = \frac{1}{2}([d\boldsymbol{\omega}, \boldsymbol{\omega}] - [\boldsymbol{\omega}, d\boldsymbol{\omega}]) = [d\boldsymbol{\omega}, \boldsymbol{\omega}] = [F^{\boldsymbol{\omega}}, \boldsymbol{\omega}].$$

Thus, $dF^{\boldsymbol{\omega}} = [F^{\boldsymbol{\omega}}, \boldsymbol{\omega}]$ and $(\xi_1, \xi_2, \xi_3|D^{\boldsymbol{\omega}} F^{\boldsymbol{\omega}}) = (\xi_1^H, \xi_2^H, \xi_3^H|[F^{\boldsymbol{\omega}}, \boldsymbol{\omega}]) = 0$, because $\boldsymbol{\omega}$ vanishes on horizontal derivations. ■

We will show now that the graded curvature $F^{\boldsymbol{\omega}}$ satisfies the second property of the connexion $\boldsymbol{\omega}$ described in Proposition 8.2. To this end, consider first $a = \delta_g$, $g \in G$. Then:

$$\begin{aligned}\Phi_g^* F^{\boldsymbol{\omega}} &= \Phi_g^*(d\boldsymbol{\omega} + \frac{1}{2}[\boldsymbol{\omega}, \boldsymbol{\omega}]) = d\Phi_g^*\boldsymbol{\omega} + \frac{1}{2}\Phi_g^*[\boldsymbol{\omega}, \boldsymbol{\omega}] \\ &= (id \otimes AD_{g^{-1}*})d\boldsymbol{\omega} + \frac{1}{2}(id \otimes AD_{g^{-1}*})[\boldsymbol{\omega}, \boldsymbol{\omega}] = (id \otimes AD_{g^{-1}*})F^{\boldsymbol{\omega}}.\end{aligned}$$

Suppose now that $a \in \mathfrak{g}$ homogeneous; thanks to Proposition 7.2, direct calculation gives:

$$\begin{aligned}(\Phi^*)_a F^{\boldsymbol{\omega}} &= \boldsymbol{L}_{(\Phi^*)_a} F^{\boldsymbol{\omega}} = \boldsymbol{L}_{(\Phi^*)_a} d\boldsymbol{\omega} + \frac{1}{2}\boldsymbol{L}_{(\Phi^*)_a}[\boldsymbol{\omega}, \boldsymbol{\omega}] \\ &= d\boldsymbol{L}_{(\Phi^*)_a}\boldsymbol{\omega} + \frac{1}{2}\big([\boldsymbol{L}_{(\Phi^*)_a}\boldsymbol{\omega}, \boldsymbol{\omega}] + [\boldsymbol{\omega}, \boldsymbol{L}_{(\Phi^*)_a}\boldsymbol{\omega}]\big) \\ &= -d(id \otimes AD_{*a})\boldsymbol{\omega} + \frac{1}{2}\big([-(id \otimes AD_{*a})\boldsymbol{\omega}, \boldsymbol{\omega}] + [\boldsymbol{\omega}, -(id \otimes AD_{*a})\boldsymbol{\omega}]\big) \\ &= -\big((id \otimes AD_{*a})d\boldsymbol{\omega} + \frac{1}{2}(id \otimes AD_{*a})[\boldsymbol{\omega}, \boldsymbol{\omega}]\big) = -(id \otimes AD_{*a})F^{\boldsymbol{\omega}},\end{aligned}$$

because $|\boldsymbol{\omega}| = (1, 0)$, $|\boldsymbol{L}_{(\Phi^*)_a}| = (0, |a|)$. We have thus proved the following:

9.4 Property. *If $a \in \mathscr{A}(G)^\circ$ is a group-like or primitive element with respect to δ_e, we have $(\Phi^*)_a F^{\boldsymbol{\omega}} = (id \otimes AD_{*s^\circ_{\mathscr{A}}(a)}) \circ F^{\boldsymbol{\omega}}$.*

We finally prove that the graded curvature provides a criterion for checking whether the horizontal distribution is involutive or not.

9.5 Theorem. *The graded horizontal distribution is involutive if and only if the graded curvature $F^{\boldsymbol{\omega}}$ of the connexion $\boldsymbol{\omega}$ is zero: $[\mathscr{H}, \mathscr{H}] \subset \mathscr{H} \Leftrightarrow F^{\boldsymbol{\omega}} = 0$.*

Proof. It is sufficient to check the involutivity of $\mathscr{H}(Y)$. If $\xi, \eta \in \mathscr{H}(Y)$ and $\mathscr{H}$ is involutive, then: $([\xi, \eta]|\boldsymbol{\omega}) = 0 \Rightarrow (\xi, \eta|d\boldsymbol{\omega}) + \frac{1}{2}(\xi, \eta|[\boldsymbol{\omega}, \boldsymbol{\omega}]) = 0 \Rightarrow (\xi, \eta|F^{\boldsymbol{\omega}}) = 0$. But $F^{\boldsymbol{\omega}}$ vanishes identically on vertical derivations by its definition (relation (9.2)); thus $F^{\boldsymbol{\omega}} = 0$.

Conversely, suppose that $F^{\boldsymbol{\omega}} = 0$. Then for each $\xi, \eta \in \mathscr{H}(Y)$, we find: $(\xi, \eta|F^{\boldsymbol{\omega}}) = 0 \Rightarrow (\xi, \eta|d\boldsymbol{\omega}) + \frac{1}{2}(\xi, \eta|[\boldsymbol{\omega}, \boldsymbol{\omega}]) = 0 \Rightarrow (\xi, \eta|d\boldsymbol{\omega}) = 0 \Rightarrow -([\xi, \eta]|\boldsymbol{\omega}) = 0$, which implies that the derivation $[\xi, \eta]$ is also horizontal, $[\xi, \eta] \in \mathscr{H}(Y)$. ■

10. Concluding remarks

Consider a graded principal bundle $(Y, \mathscr{B})$ equipped with a connection form $\boldsymbol{\omega} \in \Omega^1(Y, \mathscr{B}, \mathfrak{g})$. We know from the general theory of graded differential forms, [**15**], that there always exists an algebra morphism κ: $\Omega(Y, \mathscr{B}) \to \Omega(Y)$ defined as follows: if i: $(Y, C^\infty) \to (Y, \mathscr{B})$ is the morphism of graded manifolds determined by $\mathscr{B}(Y) \ni f \mapsto \tilde{f} \in C^\infty(Y)$ (see exact sequence (2.1)), then i^* is just κ. On the other hand, the decomposition $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ induces a canonical projection π_0: $\mathfrak{g} \to \mathfrak{g}_0$. So we have a linear map $\kappa_0 = \kappa \otimes \pi_0$: $\Omega(Y, \mathscr{B}, \mathfrak{g}) \to \Omega(Y) \otimes \mathfrak{g}_0$. Explicitly, if $\alpha = \sum_i \alpha^i \otimes e_i \in \Omega(Y, \mathscr{B}, \mathfrak{g})$, then $\kappa_0(\alpha) = \sum_i \kappa\big((\alpha^i)_0\big) \otimes (e_i)_0$, where $(\alpha^i)_0 \in \Omega(Y, \mathscr{B})_0$ and $(e_i)_0 \in \mathfrak{g}_0$ are the even elements in the development of α; moreover, we easily realize that κ_0 is not a $(\mathbf{Z} \oplus \mathbf{Z}_2)$-graded Lie algebra morphism.

As we have seen in the proof of Theorem 9.2, $\boldsymbol{\omega}$ can be decomposed as $\boldsymbol{\omega} = \boldsymbol{\omega}_0 + \boldsymbol{\omega}_1$, where $\boldsymbol{\omega}_0 \in \Omega^1(Y, \mathscr{B})_0 \otimes \mathfrak{g}_0$ and $\boldsymbol{\omega}_1 \in \Omega^1(Y, \mathscr{B})_1 \otimes \mathfrak{g}_1$. Then clearly, $\kappa_0(\boldsymbol{\omega}) = \kappa_0(\boldsymbol{\omega}_0) = \sum_i \kappa(\boldsymbol{\omega}^i) \otimes e_i$, where i labels the even elements. Using the fact that the derivations induced on $(Y, \mathscr{B})$ and (Y, C^∞) by the right actions of $(G, \mathscr{A})$ and (G, C^∞) respectively (according to Theorem 3.9) are i-related, as well as the defining properties of a graded connection form, one can prove that $\kappa_0(\boldsymbol{\omega})$ is a connection form on the ordinary principal bundle (Y, C^∞). Furthermore, the curvatures of $\boldsymbol{\omega}$ and $\kappa_0(\boldsymbol{\omega})$ are related through $\kappa_0(F^{\boldsymbol{\omega}}) = F^{\kappa_0(\boldsymbol{\omega})}$.

The previous observations suggest that the connection theory on graded principal bundles is the suitable framework for the mathematical formulation of super-gauge field theories. The fact that the graded connection $\boldsymbol{\omega}$ splits always as $\boldsymbol{\omega} = \boldsymbol{\omega}_0 + \boldsymbol{\omega}_1$ with $\kappa_0(\boldsymbol{\omega}_0)$ being a usual connection form, incorporates automatically the idea of supersymmetric partners: $\boldsymbol{\omega}_0$ corresponds to the gauge potential of an ordinary Yang-Mills theory, while $\boldsymbol{\omega}_1$ corresponds to its supersymmetric partner. A very interesting feature of this approach is that there is no graded connection form $\boldsymbol{\omega}$ for which one of the terms $\boldsymbol{\omega}_0$ or $\boldsymbol{\omega}_1$ is zero. In physics terminology, all gauge potentials have super-partners. The same is also true for the curvature $F^{\boldsymbol{\omega}}$, the super-gauge field, since $F^{\boldsymbol{\omega}} = F_0^{\boldsymbol{\omega}} + F_1^{\boldsymbol{\omega}}$, $F_0^{\boldsymbol{\omega}} \in \Omega^2(Y, \mathscr{B})_0 \otimes \mathfrak{g}_0$, $F_1^{\boldsymbol{\omega}} \in \Omega^2(Y, \mathscr{B})_1 \otimes \mathfrak{g}_1$. Thus, our approach sets the gauge potentials (resp. fields) and its supersymmetric partners on the same footing: they both "live" in a Lie

superalgebra $\mathfrak{g}$ as components of the same connection form (resp. curvature form). This is an essential difference between our approach and the standard treatment of this problem by means of DeWitt's or Roger's supermanifolds, (see [**10**] and references therein), where the connections take values in the even part of the $\mathbf{Z}_2$-graded Lie module corresponding to a Lie supergroup.

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